1. STATIC GAMES OF COMPLETE INFORMATION

1.1. What is Game Theory?

Definition 1.1.1 Game Theory is a formal way to analyze interaction among a group of rational agents who behave strategically.

This definition contains a number of important concepts, which are discussed in order:

Group: In any game there is more than one decision maker who is referred to as player. If there is a single player the game becomes a decision problem.

Interaction: What one individual player does directly affects at least one other player in the group. Otherwise the game is simple a series of independent decision problems.

Strategic: Individual players account for this interdependence.

Rational: While accounting for this interdependence each player chooses her best action. This condition can be weakened and we can assume that agents are boundedly rational. Behavioral economics analyzes decision problems in which agents behave boundedly rational. Evolutionary game theory is game theory with boundedly rational agents.

Example 1.1.2 Ten people go into a restaurant. If everyone pays for his own meal, it is a decision problem. If everyone agrees before the meal to split the bill evenly among all ten participants, it is a game.

Game theory has found numerous applications in all fields of economics:

1. Trade: Levels of imports, exports, and prices depend not only on your own tariffs but also on tariffs of other countries.

2. Labor: Internal labor market promotions like tournaments: your chances depend not only on effort but also on efforts of others.

3. Industrial Organization: Price depends not only on your output but also on the output of your competitors.


5. Political Economy: Who/what I vote for depends on what everyone else is voting for.

1.2. Strategic- (or Normal-) Form Games

Game theory can be regarded as a multi-agent decision problem. It's useful to denote first exactly what we mean by a game. Every strategic form (normal form) game has the following ingredients.

We have a nonempty, finite set $I = \{1, 2, \ldots, n\}$ of players.

The i-th player, $i \in I$, has a nonempty set of strategies – his strategy space $S_i$ – available to him, from which he can choose one strategy $s_i \in S_i$. A strategy need not refer to a single, simple, elemental action; in a game with temporal structure a strategy can be a very complex sequence of actions that depend on the histories of simple actions taken by all other players.
We will see this clearly when we learn to transform an extensive form description of a game into its strategic form. The name “strategic form” derives precisely because the present formalism ignores all this potential complexity and considers the strategies as primitives of the theory (i.e. as units which cannot be decomposed into simpler constituents). Each player has his own strategy space \( S_i \). Therefore each player has access to his own possibly unique set of strategies. We will assume that each player’s strategy space is finite. When necessary we will refer to these as pure strategies in order to distinguish them from mixed strategies, which are randomizations over pure strategies.

**Example 1.2.1** Consider a two-player game between Ann and Bob. Suppose, Ann has three actions available to her: *Left, Middle, Right*. Then her strategy space \( S_A \) is

\[
S_A = \{ \text{Left, Middle, Right} \}
\]

When she plays the game she can choose only one of these actions, so her action is either \( s_A = \text{Left} \) or \( s_A = \text{Middle} \) or \( s_A = \text{Right} \). Analogously suppose Bob has two actions available to him, *Up* and *Down*. Then his strategy space \( S_B \) is

\[
S_B = \{ \text{Up, Down} \}
\]

and he can choose either \( s_B = \text{Up} \) or \( s_B = \text{Down} \).

The outcome of the game is defined as follows: Let every player chooses one his strategy at the same time. Then we have strategies \( s_1 \in S_1, s_2 \in S_2, ..., s_n \in S_n \), which constitute an ordered \( n \)-tuple of individual strategies

\[
s = (s_1, s_2, ..., s_n).
\]

This \( n \)-dimensional vector is called a *strategy profile*. The set of all strategy profiles (the set of all possible game outcomes, or all possible ways how to play the game), called the *strategy profiles space*, is nothing but the Cartesian product of the strategy spaces \( S_i \) for each player \( i \in I \), i.e.

\[
S = S_1 \times S_2 \times ... \times S_n.
\]

**Example 1.2.1 (continued)** If Ann plays \( s_A = \text{Middle} \) and Bob chooses \( s_B = \text{Down} \), then the appropriate strategy profile is

\[
s = (\text{Middle}, \text{Down}).
\]

The space of all strategy profiles for this example is

\[
S = S_A \times S_B = \{(\text{Left, Up}), (\text{Middle, Up}), (\text{Right, Up}), (\text{Left, Down}), (\text{Middle, Down}), (\text{Right, Down})\}
\]

Sometime it is useful for player \( i \) to look at the actions chosen by all other players as a whole. We can represent such a \((n-1)\)-tuple of strategies, known as *complementary strategy profile*, by

\[
s_{-i} = (s_1, s_2, ..., s_i, ..., s_{i-1}, s_{i+1}, ..., s_n)
\]

To each player \( i \in I \) there exists a *complementary strategy profiles space* \( S_{-i} \), which is the space of all possible strategy choices \( s_{-i} \).
If we want to single out the strategy-decision problem for the particular player $i$, it is useful to write a strategy profile $s \in S$ as a combination of his strategy $s_i \in S_i$ and the complementary strategy profile $s_{-i} \in S_{-i}$ of the strategies of his opponents, i.e., we will write $s = (s_i, s_{-i})$

**Example 1.2.1 (continued)** If Ann and Bob play the strategy profile $s = (\text{Middle}, \text{Down})$, then $s_{-A} = s_B = \text{Down}$ and $s_{-B} = s_A = \text{Middle}$.

Players have preferences over the outcomes of the play. You should realize that players cannot have preferences over the actions. We can represent preferences over outcomes through a utility function, in Game Theory called payoff function. Mathematically, preferences over outcomes are defined as

$$u_i : S \rightarrow R$$

There are two basic ways how to represent the preferences over outcomes by a payoff function. If preferences can be valuated by some quantity (e.g., profit) we use this quantity. If there is no quantity connected to the individual outcomes, we simply order preferences from the most preferred one to the least preferred one and match them descending integers.

**Example 1.2.1 (continued)** We can present payoffs to Ann and Bob for each possible strategy profile in the form of the following table (payoff bi-matrix).

<table>
<thead>
<tr>
<th></th>
<th>Up</th>
<th>Down</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ann</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Left</td>
<td>2, 4</td>
<td>7, 1</td>
</tr>
<tr>
<td>Middle</td>
<td>3, 6</td>
<td>3, 6</td>
</tr>
<tr>
<td>Right</td>
<td>2, 8</td>
<td>6, 3</td>
</tr>
</tbody>
</table>

Fig. 1.1 Payoff bi-matrix

Each row corresponds to one strategy available to Ann and each column corresponds to one strategy available to Bob. Hence, Ann is called a row player, while Bob is called a column player. Each box crossing a row strategy with a column strategy corresponds to a possible game outcome and contains a pair of payoffs. The first payoff of each pair is by convention the one that Row (Ann) receives. The second is that received by Column (Bob).

To make explicit the connection between the payoff matrix and the payoff function formalism from above, we note two examples:

$$u_A(\text{Left, Down}) = 7 \quad u_B(\text{Right, Down}) = 3$$

Summarize these cornerstones of Game Theory we can formulate a definition of a strategic- (normal-) form game:

**Definition 1.2.2** A strategic- (normal-) form game $G$ is a triple

$$G = (I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I})$$

where $I = \{1, 2, \ldots, n\}$ is a set of players;

$S_i, i \in I$, are strategy spaces;

$u_i, i \in I$, are payoff functions.
In the first half of this course, we will confine ourselves to the games of complete information. A crucial assumption behind the game of complete information is that everything about the formulation of the game (i.e., the set of players, the set of strategies and the payoff functions) are all known by each player in the game. Moreover, each player knows that all players know everything about the game, and all players know that each player knows that all players know everything about the game, and so on. So, we postulate that common knowledge is the primitives of the game of complete information. We say that X is common knowledge if everyone knows X, and everyone knows that everyone knows X, and everyone knows that everyone knows that everyone knows X, ad infinitum.

A static game (simultaneous game) of complete information is probably the most fundamental game, or environment, that game theory started studying. The basic idea that a static game represents is the following:

**Step 1.** Each player simultaneously chooses an action (i.e., a strategy) from his strategy space.

**Step 2.** All chosen actions are revealed.

**Step 3.** Payoffs are distributed among the players depending on the choice of all players’ actions.

In the simplest case we assume that players choose their strategies simultaneously, and hence we call them. However, this does not require that players strictly act at the same time. All that is necessary is that each player acts without knowledge of what others have done. That is, players cannot condition their strategies on observable actions of the other players.

### 1.3. Some Important Simultaneous Games

Now, some important simultaneous games are presented.

**Game 1.3.1 Prisoners’ Dilemma**

Probably the best-known example, which has also become a base for many other situations, is called the Prisoners’ Dilemma. It’s the mother of all cooperation games. The story goes as follows: two suspects are arrested and put into different cells before the trial. The district attorney, who is pretty sure that both of the suspects are guilty but lacks enough evidence, offers them the following deal: if both of them confess and implicate the other (action *Confess*), then each will be sentenced to, say, 5 years of prison time. If one confesses and the other does not (action *Don’t Confess*), then the betrayer goes free for his cooperation with the authorities and the non-confessor is sentenced to 6 years of prison time. Finally, if neither of them confesses, then both suspects get to serve one year. Prisoners’ Dilemma can be described as a game as follows:

**Players:** Prisoner 1, Prisoner 2

**Prisoner 1’s Strategies:** Confess, Don’t Confess

**Prisoner 2’s Strategies:** Confess, Don’t Confess

**Prisoner 1’s Payoffs:**
- $u_1(Confess, Confess) = -5$
- $u_1(Confess, Don’t Confess) = 0$
- $u_1(Don’t Confess, Confess) = -6$
- $u_1(Don’t Confess, Don’t Confess) = -1$

**Prisoner 2’s Payoffs:**
- $u_2(Confess, Confess) = -5$
- $u_2(Confess, Don’t Confess) = 0$
- $u_2(Don’t Confess, Confess) = -6$
- $u_2(Don’t Confess, Don’t Confess) = -1$
Here each payoff is the number of free years that will be enjoyed by each prisoner in the next six years. This story can be compactly represented as in Figure 1.2

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Don’t Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Prisoner 1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Confess</td>
<td>-5, -5</td>
<td>0, -6</td>
</tr>
<tr>
<td>Don’t Confess</td>
<td>-6, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

Fig. 1.2 Prisoners’ Dilemma

If we rename strategy Confess as Defect and that of Don’t Confess as Coordinate, we get a typical coordination game. Here there are some examples:

1. **Arms races.** Two countries engage in an expensive arms race, which corresponds to outcome \((\text{Defect}, \text{Defect})\). They both would like to spend their money on (say) healthcare, but if one spends the money on healthcare and the other country engages in arms build-up, the weak country will get invaded.

2. **Missile defence.** Some observers interpret the missile defence initiative proposed by the administration as a Prisoner's dilemma. Country 1 (the US) can either not build a missile defence system (strategy \(\text{Cooperate}\)) or build one (strategy \(\text{Defect}\)). Country 2 (Russia) can either not build any more missiles (strategy \(\text{Cooperate}\)) or build lots more (strategy \(\text{Defect}\)). If the US does not build a missile system, and Russia does not build more missiles then both countries are fairly well off. If Russia builds more missiles and the US has no defence then the US feels very unsafe. If the US builds a missile shield, and Russia does not missiles then the US is happy but Russia feels unsafe. If the US builds missile defence and Russia builds more missiles then they are equally unsafe as in the \((\text{Cooperate}, \text{Cooperate})\) case, but they are much less well off because they both have to increase their defence budget.

**Game 1.3.2 A Pure Coordination Game**

In this game we consider two friends who want to meet in New York. Each of the two has the option to go to the Empire State building or meet at the old oak tree in Central Park. In the game players just want to be able to meet at the same spot. They don't care if they meet at the Empire State building or at Central Park. If they choose different places they are unhappy because of miscoordination.

<table>
<thead>
<tr>
<th></th>
<th>Empire State</th>
<th>Central Park</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Friend 1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Empire State</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Central Park</td>
<td>0, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

Fig. 1.3 A Pure Coordination Game

If both friends prefer to meet say in Central Park to Empire State building, the game table becomes the form:

\[
u_i(\text{Don’t Confess}, \text{Confess}) = 0
\]

\[
u_i(\text{Don’t Confess}, \text{Don’t Confess}) = -1
\]
**Friend 2**

<table>
<thead>
<tr>
<th></th>
<th>Empire State</th>
<th>Central Park</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empire State</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Central Park</td>
<td>0, 0</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

**Fig. 1.4 A Pure Coordination Game with a preferred outcome**

**Game 1.3.3 Battle of the Sexes**

This game is interesting because it is a coordination game with some elements of conflict. The idea is that a couple want to spend the evening together. The wife wants to go to the Opera, while the husband wants to go to a box fight. Each get at least some utility from going together to at least one of the venues, but each wants to go their favorite one.

We may represent this story as a two-player simultaneous game in strategic form by means of the following bi-matrix

<table>
<thead>
<tr>
<th></th>
<th><strong>Wife</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Husband</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fight</td>
<td>2, 1</td>
<td>0, 0</td>
<td></td>
</tr>
<tr>
<td>Opera</td>
<td>0, 0</td>
<td>1, 2</td>
<td></td>
</tr>
</tbody>
</table>

**Fig. 1.5 Battle of the Sexes**

Here we have two players, **Husband** and **Wife**. Both, Husband and Wife have the same strategies, Fight and Opera, and payoffs are $u_{Husband}(Fight, Fight) = 2$, $u_{Husband}(Fight, Opera) = 0$, ...

Like the Prisoners’ Dilemma, the Battle of the Sexes is also a famous example in game theory that will help us illustrate many interesting concepts later on.

**Game 1.3.4 Chicken**

This game is an anti-coordination game in which it is mutually beneficial for the players to play different strategies. The story is that two teenagers drive towards each other on a collision course: one must swerve, or both may die in the crash, but if one driver swerves but the other does not, he will be called a "chicken" None of them wants to go out of the way - whoever 'chickens' out loses his pride, while the tough guy wins. But if both stay tough, then they break their bones. If both swerve, none of them is too happy or unhappy. this terminology is most prevalent in the economics and political science. The strategic representation of this game can be as follows:

<table>
<thead>
<tr>
<th></th>
<th><strong>Teenager 2</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Teenager 1</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Straight</td>
<td>-10, -10</td>
<td>1, -1</td>
<td></td>
</tr>
<tr>
<td>Swerve</td>
<td>-1, 1</td>
<td>0, 0</td>
<td></td>
</tr>
</tbody>
</table>

**Fig. 1.6 Chicken**

In the biological literature, this game is referred to as Hawk-Dove. This version of the game imagines two players (animals) contesting an indivisible resource who can choose between two strategies, one more escalated than the other.
**Game 1.3.5 Matching Pennies**

The game is played between two players, Player 1 and Player 2. Each player has a penny and must secretly turn the penny to heads or tails. The players then reveal their choices simultaneously. If the pennies match (both heads or both tails), Player 1 receives one dollar from Player B. If the pennies do not match (one heads and one tails), Player B receives one dollar from Player A. This is an example of a zero-sum game, where one player's gain is exactly equal to the other player's loss.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>Head: 1, -1, Tail: -1, 1</td>
</tr>
<tr>
<td>Tail</td>
<td>Head: -1, 1, Tail: 1, -1</td>
</tr>
</tbody>
</table>

**Fig. 1.7 Matching Pennies**

**Game 1.3.6 The Cournot Duopoly Game**

This game has an infinite strategy space. We consider a market for a single homogenous good whose market inverse demand function is $P = D(Q)$, $Q \geq 0$ where $P$ is the price of the good and $Q$ is the quantity demanded. We assume that the function $D$ is monotonically decreasing. Suppose that there are exactly two firms producing this good. The cost function of these firms are $C_i = C_i(Q_i)$, $Q_i \geq 0$, $i = 1, 2$ where $C_i$ is a twice differentiable function defined on $R^+$ with $C_i' > 0$ and $C_i'' < 0$. Both firms choose their production levels simultaneously.

We may model the market interaction of these two firms as a two-player simultaneous game in strategic form as follows:

<table>
<thead>
<tr>
<th>Players:</th>
<th>Firm 1, Firm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1's strategies:</td>
<td>$S_1 = {Q_1</td>
</tr>
<tr>
<td>Firm 2's strategies:</td>
<td>$S_2 = {Q_2</td>
</tr>
<tr>
<td>Firm 1's payoffs:</td>
<td>$u_1(Q_1, Q_2) = D(Q_1 + Q_2)Q_1 - C_1(Q_1)$</td>
</tr>
<tr>
<td>Firm 2's payoffs:</td>
<td>$u_2(Q_1, Q_2) = D(Q_1 + Q_2)Q_2 - C_2(Q_2)$</td>
</tr>
</tbody>
</table>

### 1.4. Solving the Game. Dominance.

Now that we have found a concise way to represent games, we would like to go a step further and be able to make some prediction, or be able to prescribe behavior, about how players should (or will) play.

Let us begin with the Prisoners’ dilemma given above, and imagine that you were to advise Prisoner 1 about how to behave (say, you are his lawyer). Being a thorough person, you make the following observation for Prisoner 1: “If Prisoner 2 chooses Confess, then playing Confess gives you -5, while playing Don’t Confess gives you -6, so Confess is better. If, however, Prisoner 2 chooses Don’t Confess, then playing Confess gives you 0, while playing Don’t Confess gives you -1, so Confess, is better. It seems like Don’t Confess, is always better!” Indeed, the same analysis works for Prisoner 2, and this is the “dilemma”: each player is
better off playing Confess, regardless of his opponents actions, but this leads them both to receive payoffs of -5, while if they can only agree to both choose Don’t Confess, then they would obtain -1 each. However, left to their own devices, the players cannot resist the temptation to choose Confess. The strategy Confess is an example of a dominant strategy.

Definition 1.4.1 Given a game in strategic form, a strategy $s_i \in S_i$ (strictly) dominates strategy $\overline{s}_i \in S_i$ for player $i$, $i \in I$ if

$$u_i(s_i, s_{-i}) > u_i(\overline{s}_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$. We say that strategy $\overline{s}_i$ is (strictly) dominated by strategy $s_i$. Similarly, a strategy $s_i \in S_i$ weakly dominates strategy $\overline{s}_i \in S_i$ for player $i$, $i \in I$ if

$$u_i(s_i, s_{-i}) \geq u_i(\overline{s}_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$

and

$$u_i(s_i, s_{-i}) > u_i(\overline{s}_i, s_{-i})$$

for some $s_{-i} \in S_{-i}$

We say that strategy $\overline{s}_i$ is weakly dominated by strategy $s_i$.

Proposition 1.4.2 If player $i \in I$ is rational, he will never play a dominated strategy.

Definition 1.4.2 Given a game in strategic form, a strategy $s_i \in S_i$ is (strictly) dominant for player $i$, $i \in I$ if it (strictly) dominates every other strategy $\overline{s}_i \in S_i$. Similarly, a strategy $s_i \in S_i$ is weakly dominant for player $i$, $i \in I$ if it weakly dominates every other strategy $\overline{s}_i \in S_i$.

If player has a dominant strategy $s_i$, $s_i$ must be unique. If every player had such a wonderful strategy, then this would be a very sensible prediction for behavior. More generally,

Definition 1.4.3 The strategy profile $s^* \in S$, $s^* = (s_1^*, s_2^*, ..., s_n^*)$ is a (strict) dominant strategy equilibrium if for all players $i \in I$, $s_i^*$ is a dominant strategy.

Proposition 1.4.4 If a static game $G$ has a dominant strategy equilibrium $s^*$, then $s^*$ is the unique dominant strategy equilibrium.

The proposition is very easy to prove.

Example 1.4.5 The strategy profile (Confess,Confess) is the unique equilibrium to the Prisoners’ Dilemma game.

Dominant strategy equilibrium is quite a reasonable solution concept that does not demand an excessive amount of “rationality” from the players. It only demands the players to be (rational) optimizers, and does not require them to know that the others are rational too. Unfortunately, this concept does not work in many interesting games since the existence of a dominant strategy for all players in a game is relatively rare phenomenon. We can try something much less extreme that is based on the idea of dominance, and common knowledge of rationality, that we have imposed on the game.

This method is called Iterated Elimination of Strictly Dominated Strategies, IESDS, and works as follows: Rational players will not play strictly dominated strategies. Using our central assumptions that both the payoffs of the game and the rationality of the players is
common knowledge, all the players can “erase” the possibility that dominated strategies will be played by any player. This step may reduce the effective strategy spaces for the players, thus defining a new “smaller” game. But, it is common knowledge that in this smaller game players will not play strictly dominated strategies, and indeed there may be strategies that were not dominated in the original game, but are in the new game. Since this too is common knowledge, the process can continue and possibly reduce the set of strategies that are not dominated to make some prediction about how players will play. Formally we can define the IESDS as follows

**Definition 1.4.6** Iterated Elimination of Strictly Dominated Strategies consists of these steps:

1. **Step 1.** Define \( S^0_i = S_i \).

2. **Step 2.** Define \( S^1_i = \{ s_i \in S_i \mid -\exists s_i^* \in S_i^0 : u_i(s_i, s_{-i}) > u_i(s_i^*, s_{-i}) \forall s_{-i} \} \).

3. **Step k+1.** Define \( S^{k+1}_i = \{ s_i \in S_i^k \mid -\exists s_i^* \in S_i^k : u_i(s_i, s_{-i}) > u_i(s_i^*, s_{-i}) \forall s_{-i} \} \).

4. **Step \( \infty \).** Let \( S^\infty_i = \bigcap_{k=0}^{\infty} S^k_i \).

**Definition 1.4.7** A static game \( G \) is dominance solvable if \( S^\infty \) contains a single strategy profile.

**Proposition 1.4.8** Given a game in strategic form, if all players have strictly dominant strategies, then IESDS leads to the unique dominant strategy equilibrium; so such a game is dominance solvable.

**Example 1.4.9** Consider the following two-player game:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Left</strong></td>
<td><strong>Center</strong></td>
</tr>
<tr>
<td><strong>Up</strong></td>
<td>4, 3</td>
</tr>
<tr>
<td><strong>Middle</strong></td>
<td>2, 1</td>
</tr>
<tr>
<td><strong>Down</strong></td>
<td>3, 0</td>
</tr>
</tbody>
</table>

*Fig. 1.8* Original game

Notice first that there is no dominant strategy for Player 1 or for Player 2. Is there any (strictly) dominated strategy? Yes, *Center* is dominated by *Right* for player 2. Eliminating this strategy results in the following new game:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Left</strong></td>
<td><strong>Right</strong></td>
</tr>
<tr>
<td><strong>Up</strong></td>
<td>4, 3</td>
</tr>
<tr>
<td><strong>Middle</strong></td>
<td>2, 1</td>
</tr>
<tr>
<td><strong>Down</strong></td>
<td>3, 0</td>
</tr>
</tbody>
</table>

*Fig. 1.9* Reduced game

in which both *Middle* and *Down* are dominated by *Up* for player 1, and elimination of these two strategies yields the following trivial game:
in which player 2 has a dominant strategy, playing Left. Thus, for this example IESDS yielded the unique prediction: the strategy profile we expect these players to play is \((Up, Left)\), yielding the payoffs of \((4,3)\).

Like IESDS, the concept of Rationalizability also uses the idea that the game and rational behavior are common knowledge. Instead, however, of asking, “what would a rational player not do?”, it asks, “what might a rational player do?” The answer is that a rational player will only select strategies that are a best response to some profile of his opponents. In other words, a strategy might be played by a rational player, if he can have beliefs that would justify the play of that strategy as a best response. In turn, common knowledge of rationality implies that after employing this reasoning once, we can look at the resulting game that includes only strategies that can be a best response, and then employ this reasoning again and again, in a similar way that we did for IESDS. The solution concept of rationalizability is defined precisely by iterating this thought process.

**Definition 1.4.10** The set of strategy profiles that survive this process of rationalizability is called the set of rationalizable strategies.

We will not provide a formal definition since the introduction of mixed strategies is essential to do this.

### 1.5. Nash Equilibrium

Iterated dominance is an attractive solution concept because it only assumes that all players are rational and that it is common knowledge that every player is rational (although this might be too strong an assumption as our experiments showed). It is essentially a constructive concept – the idea is to restrict my assumptions about the strategy choices of other players by eliminating strategies one by one.

For a large class of games iterated deletion of strictly dominated strategies significantly reduces the strategy set. However, only a small class of games is solvable in this way (such as Cournot competition with linear demand curve).

Today we introduce the most important concept for solving games: Nash equilibrium. We will later show that all finite games have at least one Nash equilibrium, and that the set of Nash equilibria is a subset of the strategy profiles which survive iterated deletion. In that sense, Nash equilibrium makes stronger predictions than iterated deletion would but it is not excessively strong in the sense that it does not rule out any equilibrium play for some games.

**Definition 1.5.1** A strategy profile \(s^* \in S, s^* = (s_1^*, s_2^*, ..., s_n^*)\) is a pure strategy Nash equilibrium of strategic form game \(G\) if and only if for all player \(i, i \in I\)

\[ u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \]

for all \(s_i \in S_i\).

A pure strategy Nash equilibrium is strict if
Nash equilibrium captures the idea of equilibrium. All players know what strategy the other player is going to choose, and no player has an incentive to deviate from equilibrium play because his strategy is a best response to his belief about the other players’ strategies.

From the definition it follows that a strategy profile $s^* \in S$ is Nash equilibrium if the following condition holds for every player $i$:

$$s_i^* \in \arg \max_{s_i \in S_i} u_i(s_i, s^*)$$

Therefore, we may say that in a Nash equilibrium, each player’s choice of strategy is a best response to the strategies actually taken by his opponents. This suggests, and sometimes more useful, definition of Nash equilibrium, based on the notion of the best response correspondence.

**Definition 1.5.2** The best response correspondence of player $i$, $i \in I$ in a strategic form game $G$ is a correspondence $BR_i : S_{-i} \rightarrow S_i$ given by

$$BR_i(s_{-i}) = \{ s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(\bar{s}_i, s_{-i}) \forall \bar{s}_i \in S_i \} = \arg \max_{s_i \in S_i} u_i(s_i, s^*_i)$$

Note that for each $s_i \in S_i$, $BR_i(s_i)$ is a set that may or may not be a singleton.

**Example 1.5.3** In the game from Example 1.2.1, we have $BR_A(Up) = \{Middle\}$, $BR_A(Down) = \{Left\}$, $BR_B(Left) = \{Up\}$, $BR_B(Middle) = \{Up, Down\}$ and $BR_B(Right) = \{Up\}$.

**Proposition 1.5.4** A strategy profile $s^* \in S$, $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a pure strategy Nash equilibrium of strategic form game $G$ if and only if for all player $i$, $i \in I$

$$s_i^* \in BR_i(s^*)$$

Proposition 1.5.4 suggests a way of computing the Nash equilibria of strategic games. In particular, when the best response correspondence of the players are single-valued, then Proposition 1.5.4 tells us that all we need to do is to solve $n$ equations in $n$ unknowns to characterize the set of all Nash equilibria (once we have found $BR_i$ for all $i$, that is).

An easy way of finding Nash equilibrium in two-person strategic form games is to utilize the best response correspondences and the bimatrix representation. You simply have to mark the best response(s) of each player given the strategy choice of the other player and any strategy profile at which both players are best responding to each other is a Nash equilibrium.

**Example 1.5.3 (continued)** In the game from Example 1.2.1, we mark the best responses of both players:

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Up</td>
</tr>
<tr>
<td>Left</td>
<td>2, 4</td>
</tr>
<tr>
<td>Middle</td>
<td>3, 6</td>
</tr>
<tr>
<td>Right</td>
<td>2, 8</td>
</tr>
</tbody>
</table>

**Fig. 1.11** Game 1.2.1 with marked best responses
The set of Nash equilibria is then the set of outcomes in which both players’ payoffs are marked. In our game, there is only one such an outcome, namely strategy profile in which Ann plays Middle and Bob plays Up.

Nash equilibrium concept has been motivated in many different ways, mostly on an informal basis. We will now give a brief discussion of some of these motivations:

1. Play Prescription: Some outside party proposes a prescription of how to play the game. This prescription is stable, i.e. no player has an incentive to deviate from if he thinks the other players follow that prescription.

2. Preplay communication: There is a preplay phase in which players can communicate and agree on how to play the game. These agreements are self-enforcing.

3. Rational Introspection: A Nash equilibrium seems a reasonable way to play a game because my beliefs of what other players do are consistent with them being rational. This is a good explanation for explaining Nash equilibrium in games with the unique Nash equilibrium. However, it is less compelling for games with multiple Nash equilibria.

4. Focal Point: Social norms, or some distinctive characteristic can induce players to prefer certain strategies over others.

5. Learning Agents learn other players' strategies by playing the same game many times over.

6. Evolution: Agents are programmed to play a certain strategy and are randomly matched against each other. Assume that agents do not play Nash equilibrium initially. Occasionally 'mutations' are born, i.e. players who deviate from the majority play. If this deviation is profitable, these agents will 'multiply' at a faster rate than other agents and eventually take over. Under certain conditions, this system converges to a state where all agents play Nash equilibrium, and mutating agents cannot benefit from deviation anymore.

Remark 1. Each of these interpretations makes different assumptions about the knowledge of players. For a play prescription it is sufficient that every player is rational, and simply trusts the outside party. For rational introspection it has to be common knowledge that players are rational. For evolution players do not even have to be rational.

Remark 2. Some interpretations have less problems in dealing with multiplicity of equilibria. If we believe that Nash equilibrium arises because an outside party prescribes play for both players, then we don't have to worry about multiplicity - as long as the outside party suggests some Nash equilibrium, players have no reason to deviate. Rational introspection is much more problematic: each player can rationalize any of the multiple equilibria and therefore has no clear way to choose amongst them.

**Proposition 1.5.5** If $s^* \in S$ is either:

1. a strict dominant strategy equilibrium;
2. the unique survivor of IESDS; or
3. the unique rationalizable strategy profile;

then $s^* \in S$ is the unique Nash Equilibrium.

This proposition is simple to prove. The intuition is of course quite straightforward: we know that if there is a strict dominant strategy equilibrium then it uniquely survives IESDS, and this in turn must mean that players are playing a best response to the other players’ strategies.
Focal points are outcomes that are distinguished from others on the basis of some characteristics that are not included in the formalism of the model. Those characteristics may distinguish an outcome as a result of some psychological or social process and may even seem trivial, such as the names of the actions. Focal points may also arise due to the optimality of the strategies, and Nash equilibrium is considered focal on this basis.

Now we find Nash equilibria to the games introduced in 1.3.

**Game 1.3.1 Prisoners’ Dilemma**

Prisoners’ Dilemma has the unique Nash equilibrium – an outcome in which both players confess, i.e. the strategy profile (Confess, Confess). It is easy to check that each player can profitably deviate from every other strategy profile. For example (Don’t Confess, Don’t Confess) cannot be Nash Equilibrium because prisoner 1 would gain from playing Confess instead (as would prisoner 2).

<table>
<thead>
<tr>
<th>Prisoner 2</th>
<th>Confess</th>
<th>Don’t Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>-5, -5</td>
<td>0, -6</td>
</tr>
<tr>
<td>Don’t Confess</td>
<td>-6, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

**Fig. 1.12 Prisoners’ Dilemma**

**Game 1.3.2 A Pure Coordination Game**

This game has two Nash equilibria – (EmpireState, EmpireState) and (CentralPark, CentralPark). In both cases no player can profitably deviate. (EmpireState, CentralPark) and (CentralPark, EmpireState) cannot be Nash equilibria because both players would have an incentive to deviate. Since both equilibria lead to the same payoff to players, both players are indifferent in playing them.

<table>
<thead>
<tr>
<th>Friend 2</th>
<th>Empire State</th>
<th>Central Park</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empire State</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Central Park</td>
<td>0, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

**Fig. 1.13 A Pure Coordination Game**

In the modified game, in which both friends have preferences over places to meet, the set of Nash equilibria doesn’t change, but both players prefer Nash equilibrium (CentralPark, CentralPark) to that of (EmpireState, EmpireState) because of higher payoff. In this game, Nash equilibrium (CentralPark, CentralPark) constitutes a focal point.

<table>
<thead>
<tr>
<th>Friend 2</th>
<th>Empire State</th>
<th>Central Park</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empire State</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Central Park</td>
<td>0, 0</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

**Fig. 1.14 A Pure Coordination Game with a preferred outcome**
**Game 1.3.3 Battle of the Sexes**

(Fight, Fight) and (Opera, Opera) are both Nash equilibria of the game. The Battle of the Sexes is an interesting coordination game because players are not indifferent on which strategy to coordinate. Husband wants to watch Fight, while wife wants to go to the Opera.

<table>
<thead>
<tr>
<th></th>
<th>Fight</th>
<th>Opera</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fight</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Opera</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Fig. 1.15 Battle of the Sexes

**Game 1.3.4 Chicken**

There are two equilibria in this game, (Swerve, Straight) and (Straight, Swerve). Each player prefers not to yield to the other, the outcome where neither player yields is the worst possible one for both players.

<table>
<thead>
<tr>
<th></th>
<th>Straight</th>
<th>Swerve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight</td>
<td>-10, -10</td>
<td>1, -1</td>
</tr>
<tr>
<td>Swerve</td>
<td>-1, 1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Fig. 1.16 Chicken

**Game 1.3.5 Matching Pennies**

As you can clearly see, the method introduced above does not find a pure strategy Nash equilibrium — Given a belief that player 1 has, he always wants to match it, and given a belief that player 2 has, he would like to choose the opposite orientation for his penny. Does this mean that a Nash equilibrium fails to exist? No. It only doesn’t exist a Nash equilibrium in pure strategies. As we will see in 1.4, if we consider a richer set of possible behaviors, we will find a Nash equilibrium in mixed strategies.

<table>
<thead>
<tr>
<th></th>
<th>Head</th>
<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>Tail</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

Fig. 1.17 Matching Pennies

**Game 1.3.6 The Cournot Duopoly Game**

We consider a simplified version of the game. The market for a single homogenous good is given by inverse demand function $P = a + bQ_i$, $Q_i \geq 0$ where $P$ is the price of the good and $Q_i$ is the quantity demanded. Both firms have the same technology, so the cost function of these firms are $C_i = cQ_i$, $Q_i \geq 0$, $i = 1, 2$, and $c < a$. The payoff functions are

$$u_i(Q_i, Q_{-i}) = (a - b(Q_i + Q_{-i}))Q_i - cQ_i = (a - c - b(Q_i + Q_{-i}))Q_i$$

From the first order condition

---

15
we get the best responses of the form
\[ BR_j(Q_j) = \frac{a-c}{2b} - \frac{Q_j}{2} \]

Solving the best responses we get the Nash equilibrium of the Cournot Duopoly Game of the form \((Q_1^*, Q_2^*) = \left( \frac{a-c}{3b}, \frac{a-c}{3b} \right)\).

### 1.6. Mixed strategies

Up to now we have assumed that the only choice available to players was to pick an action from the set of available actions. In some situations a player may want to randomize between several actions. If a player chooses which action to play randomly, we say that the player is using a mixed strategy, as opposed to a pure strategy. In a pure strategy the player chooses an action for sure, whereas in a mixed strategy, he chooses a probability distribution over the set of actions available to her. In this section we will analyze the implications of allowing players to use mixed strategies.

**Definition 1.6.1** A mixed strategy for player \(i\), \(i \in I\), denoted by \(\sigma_i\), is a probability distribution over \(i\)'s set of pure strategies \(S_i\). Denote the mixed strategy space for player \(i\) by \(\Sigma_i\), where \(\sigma_i(s_{ij})\) is the probability that \(\sigma_i\) assigns to the pure strategy \(s_{ij} \in S_i\). The space of mixed strategy profiles is denoted by \(\Sigma = \Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_n\).

Thus, if player \(i\) has \(m\) pure strategies, i.e., \(S_i = \{s_{i1}, s_{i2}, \ldots s_{im}\}\), then a mixed strategy for player is a probability distribution \(\sigma_i = (\sigma_i(s_{i1}), \sigma_i(s_{i2}), \ldots \sigma_i(s_{im}))\), where \(\sigma_i(s_{ij})\) is the probability that \(\sigma_i\) assigns to the pure strategy \(s_{ij}\), i.e., \(\sigma_i(s_{ij}) \geq 0\), \(j = 1,2,\ldots, m\), and \(\sigma_i(s_{i1}) + \sigma_i(s_{i2}) + \ldots + \sigma_i(s_{im}) = 1\).

Each player’s randomization is statistically independent of those of his opponents, that is, the joint probability equals the product of individual probabilities. The payoffs to the mixed strategy profile are the expected values of the corresponding pure strategy payoffs. In all cases where we will calculate mixed strategies, the space of pure strategies will be finite so we do not run into measure-theoretic problems.

You should now see why we needed Expected Utility Theory.

**Definition 1.6.2** Player \(i\)'s payoff from a mixed strategy profile \(\sigma \in \Sigma\) in a strategic form game is
\[
    u_i(\sigma) = \sum_{s \in S} \left( \prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s)
\]

**Example 1.6.3** Recall the game from Example 1.2.1 and consider the mixed strategy profile \(\sigma = (\sigma_A, \sigma_B) = ((\sigma_A(Left), \sigma_A(Middle), \sigma_A(Right)), (\sigma_B(Up), \sigma_B(Down))) = ((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2})). In this profile, Ann plays Left with probability \(\frac{1}{3}\), Middle with probability \(\frac{1}{3}\), and Right with...
probability $\frac{5}{12}$, while Bob plays Up with probability $\frac{1}{4}$ and Down with probability $\frac{3}{4}$. There are six pure strategy profiles:

$$S = \{(Left, Up), (Middle, Up), (Right, Up), (Left, Down), (Middle, Down), (Right, Down)\}$$

that produce the six outcomes of the game. The probability of each outcome is the product of the probabilities that each player chooses the relevant strategy. For example, the probability of the pure strategy profile $(Left, Up)$ being played is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. Analogously, the probabilities of the other pure strategy profiles being played are $\text{Pr}(Middle, Up) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$, $\text{Pr}(Right, Up) = \frac{5}{12} \cdot \frac{1}{4} = \frac{5}{48}$, $\text{Pr}(Left, Down) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$, $\text{Pr}(Middle, Down) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$, and $\text{Pr}(Right, Down) = \frac{5}{12} \cdot \frac{3}{4} = \frac{5}{16}$. It is easy to verify that these sum to 1, which they must because they are probabilities of exhaustive and mutually exclusive events. Figure 1.18 shows the probability distribution over the six possible outcomes induced by the two mixed strategies.

<table>
<thead>
<tr>
<th>Bob</th>
<th>Up</th>
<th>Down</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{3}{16}$</td>
</tr>
<tr>
<td>Middle</td>
<td>$\frac{5}{12}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Right</td>
<td>$\frac{5}{12}$</td>
<td>$\frac{5}{16}$</td>
</tr>
</tbody>
</table>

![Fig. 1.18 The outcome probabilities](image)

Multiplying the payoffs by the probability of obtaining them and summing over yields to Ann an expected payoff of $\frac{1}{16} \cdot 2 + \frac{1}{12} \cdot 3 + \frac{5}{48} \cdot 2 + \frac{1}{4} \cdot 7 + \frac{1}{4} \cdot 3 + \frac{2}{16} \cdot 6 = \frac{217}{48}$. Thus, Ann’s expected payoff from the mixed strategy profile $\sigma$ as specified above is $\frac{217}{48}$. Note how we first did the multiplication term and then summed over all available pure strategy profiles, while multiplying by the utility of each. Analogously, Bob’s expected payoff from this mixed strategy profile is $\frac{1}{16} \cdot 4 + \frac{1}{12} \cdot 6 + \frac{5}{48} \cdot 8 + \frac{1}{16} \cdot 1 + \frac{1}{4} \cdot 6 + \frac{5}{16} \cdot 3 = \frac{202}{48}$.

**Definition 1.6.4** The set of player’s $i$ pure strategies to which $\sigma_i$ assigns positive probability is called support of $\sigma_i$, i.e.,

$$\text{supp}(\sigma_i) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$$

In particular, a pure strategy $s_i$ is a degenerate mixed strategy that assigns 1 to $s_i$ and 0 to all remaining pure strategies of player $i$, i.e. the support of a degenerate mixed strategy consists of a single pure strategy. A completely mixed strategy assigns positive probability to every strategy in $S_i$.

**Definition 1.6.5** A mixed strategy profile $\sigma^* \in \Sigma$, $\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ is a mixed strategy Nash equilibrium of strategic form game $G$ if and only if for all player $i$, $i \in I$

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i \in \Sigma_i$.

This definition is the natural generalization of what we defined previously as the following definition does:
**Definition 1.6.6** The best response correspondence of player \( i, i \in I \) in a strategic form game \( G \) is a correspondence \( \text{BR}_i : \Sigma_i \rightarrow \Sigma_i \) given by

\[
\text{BR}_i(\sigma_i) = \{ \sigma_i \in \Sigma_i \mid u_i(\sigma_i, \sigma_{-i}) \geq u_i(\bar{\sigma}_i, \sigma_{-i}) \forall \bar{\sigma}_i \in \Sigma_i \} \equiv \arg \max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i})
\]

**Proposition 1.6.7** A mixed strategy profile \( \sigma^* \in \Sigma, \sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*) \) is a mixed strategy Nash equilibrium of strategic form game \( G \) if and only if for all player \( i, i \in I \)

\[
\sigma_i^* \in \text{BR}_i(\sigma_{-i}^*)
\]

As before, a strategy profile is a Nash equilibrium whenever all players’ strategies are best responses to each other. For a mixed strategy to be a best response, it must put positive probabilities only on pure strategies that are best responses.

**Example 1.6.8** Consider the Matching Pennies Game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Head</td>
</tr>
<tr>
<td>Head</td>
<td>1, -1</td>
</tr>
<tr>
<td>Tail</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

Fig. 1.19 Matching Pennies

and recall that we showed that this game does not have a pure strategy Nash equilibrium. We now ask, does it have a mixed strategy Nash equilibrium? To answer this, we have to find mixed strategies for both players that are mutual best responses. To try and do this, define mixed strategies for players 1 and 2 as follows:

- let \( p \) be the probability that Player 1 plays Head, i.e., \( \sigma_1 = (p, 1 - p) \)
- let \( q \) be the probability that Player 2 plays Head, i.e., \( \sigma_2 = (q, 1 - q) \)

Now, we consider the different alternatives for Player 1 when Player 2 is believed to be playing some \( q \in [0,1] \) as follows:

\[
u_1(\text{Head}, \sigma_2) = q \cdot 1 + (1 - q) \cdot (-1) = 2q - 1
\]

\[
u_1(\text{Tail}, \sigma_2) = q \cdot (-1) + (1 - q) \cdot 1 = 1 - 2q
\]

If \( u_1(\text{Head}, \sigma_2) > u_1(\text{Tail}, \sigma_2) \) then Player 1’s best response to \( \sigma_2 \) is Head (i.e., \( p = 1 \)). This is the case when \( 2q - 1 > 1 - 2q \), or \( q > \frac{1}{2} \). If \( u_1(\text{Head}, \sigma_2) < u_1(\text{Tail}, \sigma_2) \) then Player 1’s best response to \( \sigma_2 \) is Tail (i.e., \( p = 0 \)). This is the case if \( q < \frac{1}{2} \). If \( u_1(\text{Head}, \sigma_2) = u_1(\text{Tail}, \sigma_2) \), Player 1 is indifferent between playing Head and Tail (he will play Head with any probability \( p \in [0,1] \)) because both strategies yield the same payoff. So, the best response of player 1 can be written in the form

\[
\text{BR}_1(\sigma_2) = \begin{cases} 
  p = 0 & \text{if } q < \frac{1}{2} \\
  p \in [0,1] & \text{if } q = \frac{1}{2} \\
  p = 1 & \text{if } q > \frac{1}{2} 
\end{cases}
\]

In a similar way we can calculate the payoffs of Player 2 given a mixed strategy \( \sigma_i = (p, 1 - p) \) of Player 1 to be,
\[ u_2(\sigma_1, \text{Head}) = p(1-1) + (1-p)1 = 1 - 2p \]
\[ u_2(\sigma_1, \text{Tail}) = p1 + (1-p)(-1) = 2p - 1 \]
and this implies that Player 2’s best response is
\[ BR_2(\sigma_1) \equiv BR_2(p) = \begin{cases} 
q = 1 & \text{if } p < \frac{1}{2} \\
q \in [0,1] & \text{if } p = \frac{1}{2} \\
q = 0 & \text{if } p > \frac{1}{2}
\end{cases} \]
We can see that there is indeed a pair of mixed strategies that form a Nash equilibrium, and these are precisely when \( \sigma^* = (\sigma_2^*, \sigma_1^*) = ((p,1-p),(q,1-q)) = \left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right) \) or, shortly, \( (p^*,q^*) = \left(\frac{1}{2},\frac{1}{2}\right) \)

Remember that a mixed strategy \( \sigma_i \) is a best response to \( \sigma_{-i} \) if, and only if, every pure strategy in the support of \( \sigma_i \) is itself a best response to \( \sigma_{-i} \). Otherwise player \( i \) would be able to improve his payoff by shifting probability away from any pure strategy that is not a best response to any that is.

This further implies that in a mixed strategy Nash equilibrium, where \( \sigma_i^* \) is a best response to \( \sigma_{-i}^* \) for all players \( i \), all pure strategies in the support of \( \sigma_i^* \) yield the same payoff when played against \( \sigma_{-i}^* \), and no other strategy yields a strictly higher payoff. We now use these remarks to characterize mixed strategy equilibria.

**Proposition 1.6.9** A mixed strategy profile \( \sigma^* \in \Sigma \), \( \sigma^* = (\sigma_1^*, \sigma_2^*,...,\sigma_n^*) \) is a mixed strategy Nash equilibrium of strategic form game \( G \) if and only if for all player \( i \), \( i \in I \)

1. \( u_i(s_i, \sigma_i^*) = u_i(s_j, \sigma_j^*) \) for all \( s_i, s_j \in \text{supp}(\sigma_i^*) \)
2. \( u_i(s_i, \sigma_i^*) \geq u_i(s_k, \sigma_i^*) \) for all \( s_i \in \text{supp}(\sigma_i^*) \) and \( s_k \notin \text{supp}(\sigma_i^*) \)

That is, the strategy profile \( \sigma^* \) is a mixed strategy Nash equilibrium if for every player, the payoff from any pure strategy in the support of his mixed strategy is the same, and at least as good as the payoff from any pure strategy not in the support of his mixed strategy when all other players play their mixed strategy Nash equilibrium mixed strategies. In other words, if a player is randomizing in equilibrium, he must be indifferent among all pure strategies in the support of his mixed strategy. It is easy to see why this must be the case by supposing that it must not. If he player is not indifferent, then there is at least one pure strategy in the support of his mixed strategy that yields a payoff strictly higher than some other pure strategy that is also in the support. If the player deviates to a mixed strategy that puts a higher probability on the pure strategy that yields a higher payoff, he will strictly increase his expected payoff, and thus the original mixed strategy cannot be optimal; i.e. it cannot be a strategy he uses in equilibrium.

**Proposition 1.6.10** A strictly dominated strategy is not used with positive probability in any mixed strategy equilibrium.

This means that when we are looking for mixed strategy equilibria, we can eliminate from consideration all strictly dominated strategies. It is important to note that, as in the case of pure strategies, we cannot eliminate weakly dominated strategies from consideration when finding mixed strategy equilibria (because a weakly dominated strategy can be used with positive probability in a mixed strategy Nash equilibrium).
Here we formulate five interpretations of mixed strategies:

**Deliberate Randomization.** The notion of mixed strategy might seem somewhat contrived and counter-intuitive. One (naïve) view is that playing a mixed strategy means that the player deliberately introduces randomness into his behavior. That is, a player who uses a mixed strategy commits to a randomization device which yields the various pure strategies with the probabilities specified by the mixed strategy. After all players have committed in this way, their randomization devices are operated, which produces the strategy profile. Each player then consults his randomization device and implements the pure strategy that it tells him to. This produces the outcome for the game.

This interpretation makes sense for games where players try to outguess each other (e.g. strictly competitive games, poker, and tax audits). However, it has two problems.

First, the notion of mixed strategy equilibrium does not capture the players’ motivation to introduce randomness into their behavior. This is usually done in order to influence the behavior of other players. We will rectify some of this once we start working with extensive form games, in which players move can sequentially.

Second, and perhaps more troubling, in equilibrium a player is indifferent between his mixed strategy and any other mixture of the strategies in the support of his equilibrium mixed strategies. His equilibrium mixed strategy is only one of many strategies that yield the same expected payoff given the other players’ equilibrium behavior.

**Equilibrium as a Steady State.** Osborne (and others) introduce Nash equilibrium as a steady state in an environment in which players act repeatedly and ignore any strategic link that may exist between successive interactions. In this sense, a mixed strategy represents information that players have about past interactions. For example, if 80% of past play by player 1 involved choosing strategy A and 20% involved choosing strategy B, then these frequencies form the beliefs each player can form about the future behavior of other players when they are in the role of player 1. Thus, the corresponding belief will be that player 1 plays A with probability .8 and B with probability .2. In equilibrium, the frequencies will remain constant over time, and each player’s strategy is optimal given the steady state beliefs.

**Pure Strategies in an Extended Game** Before a player selects an action, he may receive a private signal on which he can base his action. Most importantly, the player may not consciously link the signal with his action (e.g. a player may be in a particular mood which made him choose one strategy over another). This sort of thing will appear random to the other players if they (a) perceive the factors affecting the choice as irrelevant, or (b) find it too difficult or costly to determine the relationship.

The problem with this interpretation is that it is hard to accept the notion that players deliberately make choices depending on factors that do not affect the payoffs. However, since in a mixed strategy equilibrium a player is indifferent among his pure strategies in the support of the mixed strategy, it may make sense to pick one because of mood. (There are other criticisms of this interpretation)

**Pure Strategies in a Perturbed Game.** Harsanyi introduced another interpretation of mixed strategies, according to which a game is a frequently occurring situation, in which players’ preferences are subject to small random perturbations. Like in the previous section, random factors are introduced, but here they affect the payoffs. Each player observes his own preferences but not that of other players. The mixed strategy equilibrium is a summary of the frequencies with which the players choose their actions over time.
Establishing this result requires knowledge of Bayesian Games, which we will obtain later in the course. Harsanyi’s result is so elegant because even if no player makes any effort to use his pure strategies with the required probabilities, the random variations in the payoff functions induce each player to choose the pure strategies with the right frequencies. The equilibrium behavior of other players is such that a player who chooses the uniquely optimal pure strategy for each realization of his payoff function chooses his actions with the frequencies required by his equilibrium mixed strategy.

Beliefs. Other authors prefer to interpret mixed strategies as beliefs. That is, the mixed strategy profile is a profile of beliefs, in which each player’s mixed strategy is the common belief of all other players about this player’s strategies. Here, each player chooses a single strategy, not a mixed one. An equilibrium is a steady state of beliefs, not actions. This interpretation is the one we used when we defined MSNE in terms of best responses. The problem here is that each player chooses an action that is a best response to equilibrium beliefs. The set of these best responses includes every strategy in the support of the equilibrium mixed strategy (a problem similar to the one in the first interpretation).

1.7. The Fundamental Theorem

Since this theorem is such a central result in game theory, we will present a somewhat more formal version of it, along with a sketch of the proof. The following theorem due to John Nash (1950) establishes a very useful result which guarantees that the Nash equilibrium concept provides a solution for every finite game.

**Theorem 1.7.1** Every finite strategic form game has at least one mixed strategy equilibrium.

Recall that a pure strategy is a degenerate mixed strategy. This theorem does not assert the existence of an equilibrium with non-degenerate mixing. In other words, every finite game will have at least one equilibrium, in pure or mixed strategies.

The proof requires the idea of best response correspondences we discussed. However, it is moderately technical in the sense that it requires the knowledge of continuity properties of correspondences and some set theory.

**Proof.** Recall that player $i$’s best response correspondence $BR_i(\sigma_{-i})$ maps each strategy profile $\sigma$ to a set of mixed strategies that maximize player $i$’s payoff when the other players play $\sigma_{-i}$. Let $r_i = BR_i(\sigma)$ for all $\sigma \in \Sigma$ denote player $i$’s best reaction correspondence. That is, it is the set of best responses for all possible mixed strategy profiles. Define $r : \Sigma \to \Sigma$ to be the Cartesian product of the $r_i$. (That is, $r$ is the set of all possible combinations of the players best responses.) A fixed point of $r$ is a strategy profile $r^* \in r(\sigma^*)$ such that, for each player, $r_i^* \in r_i(\sigma^*)$. In other words, a fixed point of $r$ is a Nash equilibrium.

The second step involves showing that $r$ actually has a fixed point. Kakutani’s fixed point theorem establishes four conditions that together are sufficient for $r$ to have a fixed point:

1. $\Sigma$ is compact, convex, nonempty subset of a finite-dimensional Euclidean space;
2. $r(\sigma)$ is nonempty for all $\sigma$;
3. $r(\sigma)$ is convex for all $\sigma$;
4. \( r \) is upper hemi-continuous.

We must now show that \( \Sigma \) and \( r \) meet the requirements of Kakutani’s theorem. Since \( \Sigma_i \) is a simplex of dimension \( \text{card}(S_i) - 1 \) (that is, the number of pure strategies player \( i \) has less 1), it is compact, convex, and nonempty. Since the payoff functions are continuous and defined on compact sets, they attain maxima, which means \( r(\sigma) \) is nonempty for all \( \sigma \). To see the third case, note that if \( \sigma' \in r(\sigma) \) and \( \sigma'' \in r(\sigma) \) are both best response profiles, then for each player \( i \) and \( \alpha \in (0,1) \),

\[
    u_i(\alpha \sigma'_i + (1 - \alpha) \sigma''_i, \sigma_{-i}) = \alpha u_i(\sigma'_i, \sigma_{-i}) + (1 - \alpha) u_i(\sigma''_i, \sigma_{-i})
\]

that is, if both \( \sigma'_i \) and \( \sigma''_i \) are best responses for player \( i \) to \( \sigma_{-i} \), then so is their weighted average. Thus, the third condition is satisfied. The fourth condition requires sequences but the intuition is that if it were violated, then at least one player will have a mixed strategy that yields a payoff that is strictly better than the one in the best response correspondence, a contradiction.

Thus, all conditions of Kakutani’s fixed point theorem are satisfied, and the best reaction correspondence has a fixed point. Hence, every finite game has at least one Nash equilibrium.

Somewhat stronger results have been obtained for other types of games (e.g. games with uncountable number of actions). Generally, if the strategy spaces and payoff functions are well-behaved (that is, strategy sets are nonempty compact subset of a metric space, and payoff functions are continuous), then Nash equilibrium exists. Most often, some games may not have a Nash equilibrium because the payoff functions are discontinuous (and so the best reply correspondences may actually be empty).

Note that some of the games we have analyzed so far do not meet the requirements of the proof (e.g. games with continuous strategy spaces), yet they have Nash equilibria. This means that Nash’s Theorem provides sufficient, but not necessary, conditions for the existence of equilibrium. There are many games that do not satisfy the conditions of the Theorem but that have Nash equilibrium solutions.
2. DYNAMIC GAMES OF COMPLETE INFORMATION

As we have seen, the strategic form representation is a very general way of putting a formal structure on strategic situations, thus allowing us to analyze the game and reach some conclusions about what will result from the particular situation at hand. However, one obvious drawback of the strategic form is its difficulty in capturing time. That is, there is a sense in which players’ strategy sets correspond to what they can do, and how the combination of their actions affect each others payoffs, but how is the order of moves captured? More importantly, if there is a well-defined order of moves, will this have an affect on what we would label as a reasonable prediction of our model?

2.1. Extensive form Games

The concept and formalization of the extensive form game is meant to formally capture situations with sequential moves of players, and to allow for the knowledge of some players, when it is their turn to move, to depend on the previously made choices of other players. As with the strategic form, two elements must to be part of any game’s representation:

1. Set of players \( I = \{1, 2, ..., n\} \)

2. Players’ payoffs as a function of actions \( \{u_i\}_{i \in I} \)

To capture the idea of sequential play, we need to expand the rather simple notion of pure strategy sets to a more complex organization of actions. We will do this by introducing two parts to actions:

3. Order of moves

4. Choices players have when they can move

It may be that some information is revealed as the game proceeds, while other information is not. To represent this we need to be precise about the unfolding of information and knowledge:

5. Knowledge players have when they can move

To add a final component, it may be that before certain players move, some random event will happen that is not due to another players action. We will call these moves of nature, and we can think of nature as a player that has a fixed strategy, and is not strategic. Nature will be represented by:

6. A probability distribution over exogenous events

Finally, to be able to analyze these situations with the tools and concepts we have already been introduced to, we need to add the final and familiar requirement:

7. Points 1 – 6 above are common knowledge

This set up, 1 – 7 above, seems indeed to capture all of what we would expect to need in order to represent the sequential situations we want to represent. The question is, what formal notation will be used to put this all together? For this, we introduce the idea of a game tree.

The game tree is, as its name may imply, a figurative way to represent the unfolding nature of the extensive form game. Consider, for example, the following modification of Battle of
the Sexes Game. Husband first chooses whether go to box fight (Fight) or to opera (Opera). Then Wife, observing Husband’s action, also chooses whether go to fight or opera. A simple way to represent this may be with the graph depicted in Figure 2.1.

![Figure 2.1](image)

**Fig. 2.1** A sequential version of Battle of the Sexes

**Definition 2.1.1** A game tree is a tuple \((X, \phi)\), where \(X\) is a finite collection of nodes \(x \in X\) and \(\phi\) is the precedence relation where \(x \phi x'\) means that “\(x\) precedes \(x'\)” or “\(x'\) succeeds \(x\)”.

This relation is transitive and asymmetric, and thus constitutes a partial order. It is not a complete order because two nodes may not be comparable. This rules out cycles where the game may go from node \(x\) to a node \(x'\), from \(x'\) back to \(x\). In addition, we require that each node \(x\) has exactly one immediate predecessor, that is, one node \(x' \phi x\) such that \(x'' \phi x\) and \(x'' \neq x'\) implies \(x' \phi x''\) or \(x'' \phi x'\). Thus, if \(x'\) and \(x''\) are both predecessors of \(x\), then either \(x'\) is before \(x''\) or \(x''\) is before \(x'\).

This definition is quite a mouthful, but it formally captures the “physical” structure of a game tree, ignoring the actions of players and what they know when they move. Every node (point in the game) can be reached as a consequence of actions that were chosen at the node that precedes it. The root is the beginning of the game, and a terminal node is one of the many ways in which the game can end, causing payoffs to be distributed. That is, payoffs for players \(i \in I\) are given over terminal nodes: \(u_i : Z \rightarrow R^+\), where \(u_i(z)\) is \(i\)’s payoff if terminal node \(z\) is reached.

**Example 2.1.2** Consider the sequential version of Battle of the Sexes Game as depicted in Figure 2.1. In this two-player game (Husband, Wife), \(x_0\) is the root at which the game begins with Husband having two choices. His action determines whether Wife will get to play at node \(x_1\) or at node \(x_2\) and at each of these Wife has two choices that all end in termination of the game. The set of terminal nodes in this game is \(Z = \{z_1, z_2, z_3, z_4\}\).

Notice, however, that in Figure 2.1 we have a particular order of play: first Husband, then Wife. How did we assign players to nodes, when this was not part of the definition? Indeed, as noted earlier, the definition is not complete, and the following adds the way in which players are “injected” into the tree:
**Definition 2.1.3** The order of players is given by a function from non-terminal nodes to the set of players, \( \tau : \mathcal{X} \setminus \mathcal{Z} \to \mathcal{I} \), which identifies a player \( \tau(x) \) for each \( x \in \mathcal{X} \setminus \mathcal{Z} \). The set of actions that are possible at node \( x \) are denoted by \( A(x) \).

**Example 2.1.2 (continued)** In the game three in Figure 2.1 we have the following values of the identification function \( \tau(x_0) = Husband \) and \( \tau(x_1) = \tau(x_2) = Wife \). The appropriate sets of feasible actions are \( A(x_0) = A(x_1) = A(x_2) = \{Fight, Opera\} \)

There is still one missing component: how do we describe the knowledge of each player when he moves? It seems implicit if each player knows what happened before he moves. But it might be that a player needs to make his move without knowing what another player did before.

**Definition 2.1.4** Every node \( x \) has an information set \( h(x) \) that partitions the nodes of the game. If \( x \neq x' \) and if \( x' \in h(x) \), then the player who moves at \( x \) does not know whether he is at \( x \) or \( x' \).

For example, consider the original Battle of the Sexes Game, when both, Husband and Wife choose their action simultaneously. We may depict this situation as follows:

![Game Tree of static Battle of the Sexes Game](image)

**Fig. 2.2** Game Tree of static Battle of the Sexes Game

How can we use the graphical representation of the tree to distinguish whether a player knows where he is or not? To do this, we draw “ellipses” to denote information sets. For example, in Figure 2.2 Wife cannot distinguish between \( x_1 \) and \( x_2 \), so that \( h(x_1) = h(x_2) = \{x_1, x_2\} \) and both nodes are encircled together to denote this information for Wife.

Now we have a complete representation of the extensive form game.

**Definition 2.1.5** An extensive form game \( \Gamma \) is a tuple

\[
\Gamma = (I, (X, \phi), \tau, (X, \tau), H, u)
\]

where \( I = \{1, 2, ..., n\} \) is a set of players;
\( (X, \tau) \) is a tree;
Recall that we defined complete information previously in chapter 1 as the situation in which each player $i$ knows the payoff function of each $j \in I$, and this is common knowledge. This definition sufficed for the strategic form representation. For extensive form games, however, it is useful to distinguish between two different types of complete information games:

**Definition 2.1.6** A game in which every information set $h(x)$ is a singleton is called a game of perfect information. A game in which some information sets contain several nodes is called a game of imperfect information.

That is, in a game of perfect information every player knows exactly where he is in the game, while in a game of (complete but) imperfect information some players do not know where they are. Notice, therefore, that simultaneous move games fall into this second category.

In the strategic form game it was quite easy to define a strategy for a player: a pure strategy was some element from his set of actions, $S_i$, and a mixed strategy was some probability distribution over these actions. It is very easy to extend this idea to extensive form games as follows: A strategy is a complete contingent plan of action. That is, a strategy in an extensive form game is a plan that specifies the action chosen by the player for every history after which it is his turn to move, that is, at each of his information sets. This is a bit counter-intuitive because it means that the strategy must specify moves at information sets that might never be reached because of actions specified by the player’s strategy at earlier information sets.

**Definition 2.1.7** A pure strategy for player $i$, $i \in I$, in an extensive form game $\Gamma$ is a function $s_i : H \rightarrow A$ such that $s_{i(h)}(h) \in A_{i(h)}(h)$ for all $h \in H$.

**Example 2.1.2 (continued)** Since Husband has the unique node in which he chooses his action; his strategy space takes the form $S_h = \{\text{Fight}, \text{Opera}\}$. On the other hand, Wife has two nodes, $x_1$ – after Husband’s choice Fight and $x_2$ – after Opera, so her strategy space is of the form $S_w = \{\text{(Fight,Fight), (Fight,Opera), (Opera,Fight), (Opera,Opera)}\}$.

Once we have figured out what pure strategies are, the definition of mixed strategies follows immediately:

**Definition 2.1.8** A mixed strategy for player $i$, $i \in I$, in an extensive form game $\Gamma$ is a probability distribution over his pure strategies.

How do we interpret a mixed strategy? Exactly in the same way that it applies to the strategic form: a player randomly chooses between all his complete plans of play, and once a particular plan is selected the player follows it. However, notice that this interpretation takes away some of the dynamic flavor that we set out to capture with extensive form games. Namely, when a mixed strategy is used, the player selects a plan and then follows a particular pure strategy. This was sensible for strategic form games since there it was a once-and-for-all choice. In a game tree, however, the player may want to randomize at some nodes,
independently of what he did in earlier nodes where he played. This cannot be captured by mixed strategies as defined above.

**Definition 2.1.9** A *behavioral strategy* for player $i$, $i \in I$, in an extensive form game $\Gamma$ is a set of probability distributions over sets of feasible actions at all his information sets.

One can argue, that a behavioral strategy is more loyal to the dynamic nature of the extensive form game. When using such a strategy, a player mixes between his actions whenever he is called to play. This differs from a mixed strategy, in which a player mixes before playing the game, but then is loyal to the selected pure strategy.

### 2.2. Strategic form Representation of Extensive form Game

Consider the two variants of the battle of the sexes given above in figures 2.1 and 2.2. The latter is equivalent to the original game we analyzed in strategic form, and indeed, can be translated immediately into the strategic form as follows:

<table>
<thead>
<tr>
<th>Husband</th>
<th>Wife</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fight</td>
</tr>
<tr>
<td>Fight</td>
<td>2, 1</td>
</tr>
<tr>
<td>Opera</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

**Fig. 2.3** Strategic form of the game in Fig. 2.2

In fact, any extensive form game can be turned into a strategic form game that is given by the set of players, the set of derived pure strategies, and the payoffs resulting from the actual play of any specified profile of strategies. If there are two players and finite strategy sets, the game can be represented by a bi-matrix as we have discussed in the previous section on strategic form games. For example, take the game depicted in figure 2.1. This game can be represented by a $2 \times 4$ bi-matrix as follows:

<table>
<thead>
<tr>
<th>Husband</th>
<th>Wife</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fight,</td>
<td>Fight,</td>
<td>Opera</td>
<td>Opera</td>
</tr>
<tr>
<td></td>
<td>Fight</td>
<td>Opera</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fight</td>
<td>2, 1</td>
<td>2, 1</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>Opera</td>
<td>0, 0</td>
<td>1, 2</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

**Fig. 2.4** Strategic form of the game in Fig. 2.1

As this bi-matrix demonstrates, each of the four payoffs in the original extensive form game is replicated twice. This happens because for a certain choice of Husband, two pure strategies of Wife are equivalent. For example, if Husband plays *Fight*, then only the first component of Wife’s strategy matters, so that (*Fight*, *Fight*) and (*Fight*, *Opera*) yield the same outcome, as do (*Opera*, *Fight*) and (*Opera*, *Opera*). If, however, Husband plays *Opera*, then only the second component of Wife’s strategy matters. Clearly, this exercise of transforming extensive form games into the strategic form misses the dynamic nature of the extensive form game. Why, then, would we be interested in this exercise? It turns out to be very useful to find the Nash equilibria of the original extensive form game.
Definition 2.2.1 An associated game to the extensive form game \( \Gamma \) is the strategic form game \( G^* = (I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I}) \) where

- \( I \) is the set of players in the game \( \Gamma \);
- \( S_i, i \in I \), are all players’ strategy spaces in the game \( \Gamma \);
- \( u_i, i \in I \), are all players payoff functions derived from the collection of all players’ payoff functions in the game \( \Gamma \).

2.3. Nash Equilibrium of an Extensive Form Game

Definition 2.3.1 The set of all Nash equilibria of an extensive form game coincides with the set of all Nash equilibria of the associated game to it.

This is a convenient feature of the strategic form representation of an extensive form game: it will immediately reveal all the pure strategies of each player, and in turn will lead us to easily see the pure strategy profiles that are Nash equilibria. (That is, easy in matrix games, and will require a bit more work for other games by finding the best response correspondences.).

Example 2.3.2 The sequential form of Battle of the Sexes in Figure 2.1 has three pure strategy Nash equilibria: \((\text{Fight, Fight, Fight})\), \((\text{Fight, Fight, Opera})\), and \((\text{Opera, Opera, Opera})\) (see Figure 2.5).

<table>
<thead>
<tr>
<th>Husband</th>
<th>Wife</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fight</td>
<td>Fight, Fight</td>
</tr>
<tr>
<td>Fight</td>
<td>2, 1</td>
</tr>
<tr>
<td>Opera</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Fig. 2.5 Pure strategy Nash equilibria of the game in Fig. 2.1

In the extensive form, any Nash equilibrium is not only a prediction of the outcomes through the terminal nodes, but also a prediction about the path of play, also called the equilibrium path.

Consider again the extensive and strategic forms of the sequential (perfect information) Battle of the Sexes. Now ask yourself: is the Nash equilibrium \((\text{Opera, Opera, Opera})\) a reasonable prediction about the rational choice of Husband? By the definition of Nash equilibrium it is: \((\text{Opera, Opera})\) is a best response to \(\text{Opera}\), and vice versa. However, this implies that if, for some unexpected reason, Husband would suddenly choose to play \(\text{Fight}\), then Wife would not respond optimally: her strategy \((\text{Opera, Opera})\) commits her to choose \(f\) even though \(o\) would yield her a higher utility.

This is precisely the kind of “action” we cannot get in the strategic form representation because all choices are made simultaneously, and thus beliefs can never be challenged. Clearly, we would expect rational players to play optimally in response to their beliefs whenever they are called to move. This requirement will put more constraints on what we call “rational behavior”, since we should expect players to be sequentially rational, which is the focus of this chapter.
This argument would suggest that of the three Nash equilibria in this game, two seem somewhat unappealing. Namely, the equilibria \((\text{Fight}, (\text{Fight}, \text{Fight}))\) and \((\text{Opera}, (\text{Opera}, \text{Opera}))\) have Wife commit to a strategy that, despite being a best response to Husband’s strategy, would not have been optimal were Husband to deviate from his strategy and cause the game to move off the equilibrium path. In what follows, we will set up some structure that will result in more refined predictions for dynamic games. These will indeed rule out such equilibria, and as will become clear later will only admit the equilibrium \((\text{Fight}, (\text{Fight}, \text{Opera}))\) as the unique equilibrium that survives the more stringent structure.

Example 2.3.2 We consider Stackelberg Duopoly Game on the market for a single homogenous good is given by inverse demand function \(P = a + Q\), \(Q \geq 0\) where \(P\) is the price of the good and \(Q = Q_1 + Q_2\) is the aggregate quantity. First, Firm 1 chooses \(Q_1 \in [0, \frac{a}{\pi}]\), Firm 2 observes Firm 1’s choice and then chooses \(Q_2 \in [0, \frac{a}{\pi}]\). or simplicity, we consider that both firms have no cost. As in Cournot Duopoly Game the payoff functions are \(u_i(Q_1, Q_2) = (a - (Q_1 + Q_2))Q_i\).

Claim: For any \(Q_i \in [0, \frac{a}{\pi}]\) the game has a Nash equilibrium in which Firm 1 produces \(Q_1\).

Sketch of the Proof. Consider the following strategies:

\[
\begin{align*}
s_1 &= \overline{Q}_1 \\
s_2 &= \begin{cases} a - \overline{Q}_1, & \text{if } Q_1 = \overline{Q}_1 \\ \frac{a - \overline{Q}_1}{2}, & \text{if } Q_1 \neq \overline{Q}_1 \end{cases}
\end{align*}
\]

In words: Firm 2 floods the market such that the price drops to zero if Firm 1 does not choose \(\overline{Q}_1\). It is easy to see that these strategies form a Nash equilibrium. Firm 1 can only do worse by deviating since profits are zero if Firm 2 floods the market. Firm 2 plays a best response to \(\overline{Q}_1\) and therefore won't deviate either.

Note, that in this game things are even worse. Unlike the Cournot game where we got the unique equilibrium we now have a continuum of equilibria.

2.4. Sequential Rationality and Backward Induction

To address the critique that we posed in the previous example about the equilibria \((\text{Fight}, (\text{Fight}, \text{Fight}))\) and \((\text{Opera}, (\text{Opera}, \text{Opera}))\) we will directly criticize the behavior of Wife in the event that Husband did not follow his strategy.

Definition 2.4.1 A player in an extensive form game is sequentially rational, if he uses strategy that is optimal at every node in the game tree.

We call this principle sequential rationality, since it implies that players are playing rationally at every stage in the sequence of play.

Going back to the game above, we ask: what should Wife do in each of her information sets? The answer is obvious, if Husband played \text{Fight}, then Wife should play \text{Fight}, and if Husband played \text{Opera}, then Wife should play \text{Opera}. Any other strategy should not be played in each of these information sets, which implies that Wife should be playing the pure strategy of.
Now move back to the root of the game where Husband has to choose between \textit{Fight} and \textit{Opera}. Taking into account the sequential rationality of Wife, Husband should conclude that playing \textit{Fight} will result in the payoffs $(2, 1)$ while playing \textit{Opera} will result in the payoffs $(1, 2)$. Now applying sequential rationality to Husband implies that Husband, who is correctly predicting the behavior of Wife, should choose \textit{Fight}, and the unique prediction from this process is the path of play \textit{Fight} followed by \textit{Fight}.

Furthermore, the process predicts what would happen if players deviate from the path of play: if Husband chooses \textit{Opera} then Wife will choose \textit{Opera}. We conclude that the Nash equilibrium $(\textit{Fight}, (\textit{Fight}, \textit{Opera}))$ uniquely survives this procedure.

This type of procedure, which starts at nodes that precede only terminal nodes at the end of the game and moves backward, is known as backward induction. It turns out, as the example above suggests, that when we apply this procedure to finite games of perfect information, then we will get a specification of strategies for each player that are sequentially rational. By finite we mean that the game has a finite number of sequences, after which it ends.

In 1913 Zermelo proved that chess has an optimal solution. He reasoned as follows. Since chess is a finite game (it has quite a few moves, but they are not infinite), this means that it has a set of penultimate nodes. That is, nodes whose immediate successors are terminal nodes. The optimal strategy specifies that the player who can move at each of these nodes chooses the move that yields him the highest payoff (in case of a tie he makes an arbitrary selection). Now, the optimal strategies specify that the player who moves at the nodes whose immediate successors are the penultimate nodes chooses the action which maximizes his payoff over the feasible successors given that the other player moves there in the way we just specified. We continue doing so until we reach the beginning of the tree. When we are done, we will have specified an optimal strategy for each player.

These strategies constitute a Nash equilibrium because each player’s strategy is optimal given the other player’s strategy. In fact, these strategies also meet the stronger requirements of subgame perfection, which we will examine in the next section. (Kuhn’s paper provides a proof that any finite extensive form game has an equilibrium in pure strategies. It was also in this paper that he distinguished between mixed and behavior strategies for extensive form games.) Hence the following result:
Theorem 2.4.2 (Zermelo 1913; Kuhn 1953). A finite game of perfect information has a pure strategy Nash equilibrium.

Backward induction can be applied to any finite extensive form game of perfect information, and will result in a sequentially rational Nash equilibrium. Furthermore, if no two terminal nodes prescribe the same payoffs to any player, this procedure will result in the unique sequentially rational Nash equilibrium. The backward induction applied to the sequential Battle of the Sexes Game is in Figure 2.6.

2.5. Subgame Perfect Nash Equilibrium

We saw that Zermelo’s theorem is useful in helping us identify sequentially rational Nash equilibria for a large class of games. However, this is a solution procedure, not a solution concept. The reason is that this procedure only applies to finite games of perfect information, whereas a solution concept such as Nash equilibrium applies to all extensive form games (by the fact that they can be represented as a strategic form game).

Our next goal is to find a natural way to extend the ideas of sequential rationality to games that do not satisfy the perfect information structure. Intuitively, when a player has non-degenerate information sets, we may not be able to identify his best action since it may depend on which node he is at, which itself depends on the actions of players that moved before him. Thus, we will have to consider a solution that looks at sequential rationality of the game with these types of dependencies.

Definition 2.5.1 A subgame $\Gamma_x$ of an extensive form game $\Gamma$ is a part of the game tree such that

1. it starts at a single decision node $x$;
2. it contains every successors of $x$;
3. if it contains a node in an information set, then it contains all nodes in that information set.

It is convenient to treat the entire game as a subgame. all the other subgames call proper subgames. Given a subgame $\Gamma_x$, let us denote the restriction of a strategy $s_i$ to that subgame by $s_i \mid \Gamma_x$.

Definition 2.5.2 A strategy profile $s^* \in S$ in an extensive form game $\Gamma$ is a subgame perfect Nash equilibrium, if for every subgame $\Gamma_x$ of $\Gamma$, $s^* \mid \Gamma_x$ induces a Nash equilibrium in $\Gamma_x$.

Notice that by the definition of a subgame perfect Nash equilibrium, every subgame perfect Nash equilibrium is a Nash equilibrium. However, not all Nash equilibria are necessarily subgame perfect, implying that subgame perfection refines the set of Nash equilibria, yielding more refined predictions on behavior. To see this consider the sequential Battle of the Sexes Game in Figure 2.1, in which we have identified three pure strategy Nash equilibria, $(\text{Fight},(\text{Fight},\text{Fight}))$, $(\text{Fight},(\text{Fight},\text{Opera}))$, and $(\text{Opera},(\text{Opera},\text{Opera}))$. Of these three, only $(\text{Fight},(\text{Fight},\text{Opera}))$ is the unique subgame-perfect equilibrium.
This follows because in the subgame beginning at \( x_1 \), the only Nash equilibrium is Wife choosing *Fight*, since she is the only player in that subgame and she must choose a best response to Husband’s choice *Fight*. Similarly, in the subgame beginning at \( x_2 \), the only Nash equilibrium is Wife choosing *Opera*. Thus, of the three Nash equilibria of the whole game, only \((Fight, (Fight, Opera))\) satisfies the condition that its restriction is a Nash equilibrium for every proper subgame of the whole game.

**Example 2.5.3** To see the application in a game with continuous strategy sets, consider the Stackelberg Duopoly Game given in Example 2.3.2. We claim that the unique subgame perfect Nash equilibrium is

\[
\left( Q^*_1, Q^*_2(Q^*_1) \right) = \left( \frac{a}{2}, \frac{a - Q^*_1}{2} \right) = \left( \frac{a}{2}, \frac{a}{4} \right)
\]

The proof is as follows. A subgame perfect equilibrium must be a Nash equilibrium in the subgame after Firm 1 has chosen \( Q_1 \). This is a one-player game so Nash equilibrium is equivalent to Firm 2 maximizing its payoff, i.e.

\[
Q^*_2(Q_1) = \arg \max_{Q_2} (a - (Q_1 + Q_2)Q_2)
\]

This implies that \( Q^*_2(Q_1) = \frac{a - Q_1}{2} \). Equivalently, Firm 2 plays on its best response curve. A subgame perfect Nash equilibrium must also be a Nash equilibrium in the whole game, so \( Q_1 \) is a best response to \( Q_2 \):

\[
u_1(Q_1, Q_2^*) = (a - (Q_1 + Q_2(Q_1)))Q_1
\]

Maximizing the Firm 1’s payoff function we get \( Q_1 = \frac{a}{2} \) and \( Q_2 = \frac{a}{4} \).

**Example 2.5.4** *(The Ultimatum Game)* Two players want to split a pie of size \( \pi > 0 \). Player 1 offers a division \( x \in [0, \pi] \) according to which his share is \( x \) and player 2’s share is \( \pi - x \). If player 2 accepts this offer, the pie is divided accordingly. If player 2 rejects this offer, neither player receives anything.

In this game, player 1 has a continuum of action available at the initial node, while player 2 has only two actions. (The continuum of actions ranges from offering 0 to offering the entire pie) When player 1 makes some offer, player 2 can only accept or reject it. There is an infinite number of subgames following a proposal by player 1. Each history is uniquely identified by the proposal, \( x \). In all subgames with \( x < \pi \) player 2’s optimal action is to accept because doing so yields a strictly positive payoff that is higher than 0, which is what he would get by rejecting. In the subgame following the history \( x = \pi \), however, player 2 is indifferent between accepting and rejecting. So in a subgame perfect Nash equilibrium player 2’s strategy either accepts all offers (including \( x = \pi \)) or accepts all offers \( x < \pi \) and rejects \( x = \pi \).

Given these strategies, consider player 1’s optimal strategy. We have to find player 1’s optimal offer for every SPE strategy of player 2. If player 2 accepts all offers, then player 1’s optimal offer is \( x = \pi \) because this yields the highest payoff. If player 2 rejects \( x = \pi \) but accepts all other offers, there is no optimal offer for player 1! To see this, suppose player 1 offered some \( x < \pi \), which player 2 accepts. But because player 2 accepts all \( x < \pi \), player 1
can improve his payoff by offering some \( x' \) such that \( x < x' < \pi \), which player 2 will also accept but which yields player 1 a strictly better payoff.

Therefore, the ultimatum game has a unique subgame perfect Nash equilibrium, in which player 1 offers \( x = \pi \) and player 2 accepts all offers. The outcome is that player 1 gets to keep the entire pie, while player 2’s payoff is zero.

This one-sided result comes for two reasons. First, player 2 is not allowed to make any counteroffers. If we relaxed this assumption, the subgame perfect Nash equilibrium will be different. (In fact, in the next section we will analyze a very general bargaining model.) Second, the reason player 1 does not have an optimal proposal when player 2 accepts all offers has to do with him being able to always do a little better by offering to keep slightly more. Because the pie is perfectly divisible, there is nothing to pin the offers. However, making the pie discrete (e.g. by slicing it into \( n \) equal pieces and then bargaining over the number of pieces each player gets to keep) will change this as well.

### 2.6. The Rubinstein Bargaining Model

There are at least two basic ways one can approach the bargaining problem. (The bargaining problem refers to how people would divide some finite benefit among themselves.) Nash initiated the axiomatic approach with his Nash Bargaining Solution (he did not call it that, of course). This involves postulating some desirable characteristics that the distribution must meet and then determining whether there is a solution that meets these requirements. This is very prominent in economics but we will not deal with it here.

Instead, we will look at strategic bargaining. Unlike the axiomatic solution, this approach involves specifying the bargaining protocol (i.e. who gets to make offers, who gets to respond to offers, and when) and then solving the resulting extensive form game.

People began analyzing simple two-stage games (e.g. ultimatum game where one player makes an offer and the other gets to accept or reject it) to gain insight into the dynamics of bargaining. Slowly they moved to more complicated settings where one player makes all the offers while the other accepts or rejects, with no limit to the number of offers that can be made. The most attractive protocol is the alternating-offers protocol where players take turns making offers and responding to the other player’s last offer.

The alternating-offers game was made famous by Ariel Rubinstein in 1982 when he published a paper showing that while this game has infinitely many Nash equilibria (with any division supportable in equilibrium), it had a unique subgame perfect Nash equilibrium! Now this is a great result and since it is the foundation of most contemporary literature on strategic bargaining, we will explore it in some detail.

**Example 2.6.1** (The Basic Alternating-Offers Model). Two players, A and B, bargain over a partition of a pie of size \( \pi > 0 \) according to the following procedure. At time \( t = 0 \) player A makes an offer to player B about how to partition the pie. If player B accepts the offer, then an agreement is made and they divide the pie accordingly, ending the game. If player B rejects the offer, then he makes a counteroffer at time \( t = 1 \). If the counteroffer is accepted by player A, the players divide the pie accordingly and the game ends. If player A rejects the offer, then he makes a counter-counteroffer at time \( t = 2 \). This process of alternating offers and counteroffers continues until some player accepts an offer.
To make the above a little more precise, we describe the model formally. Two players, A and B, make offers at discrete points in time indexed by $t = (0, 1, 2, \ldots)$. At time $t$ when $t$ is even (i.e., $t = 2, 4, 6, \ldots$) player A offers $x \in [0, \pi]$ where $x$ is the share of the pie A would keep and $\pi - x$ is the share B would keep in case of an agreement. If B accepts the offer, the division of the pie is $(x, \pi - x)$. If player B rejects the offer, then at time $t + 1$ he makes a counteroffer $y \in [0, \pi]$. If player A accepts the offer, the division $(\pi - y, y)$ obtains. Generally, we will specify a proposal as an ordered pair, with the first number representing player A’s share. Since this share uniquely determines player B’s share (and vice versa) each proposal can be uniquely characterized by the share the proposer offers to keep for himself.

The players discount the future with a common discount factor $\delta \in (0, 1)$. Hence the payoffs are as follows. While players disagree, neither receives anything (which means that if they perpetually disagree then each player’s payoff is zero). If some player agrees on a partition $(x, \pi - x)$ at some time $t$, player A’s payoff is $x \delta^t$ and player B’s payoff is $\delta^t (\pi - x)$.

This completes the formal description of the game.

Let’s find the Nash equilibria in pure strategies for this game. Actually, we cannot find all Nash equilibria because there’s an infinite number of those. What we can do, however, is characterize the payoffs that players can get in equilibrium.

**Claim 2.6.2** Any division of the pie can be supported in some Nash equilibrium.

**Proof.** To see this, consider the strategies where player A demands $x \in [0, \pi]$ in the first period, then $\pi$ in each subsequent period where he gets to make an offer, and always rejects all offers. This is a valid strategy for the bargaining game. Given this strategy, player B does strictly better by accepting $x$ instead of rejecting forever, so she accepts the initial offer and rejects all subsequent offers. Given that B accepts the offer, player A’s strategy is optimal.

The problem, of course, is that Nash equilibrium requires strategies to be mutually best responses only along the equilibrium path. It is just not reasonable to suppose that player A can credibly commit to rejecting all offers regardless of what player B does. To see this, suppose at some time $t > 0$, player B offers $y < \pi$ to player A. According to the Nash equilibrium strategy, player A would reject this (and all subsequent offers), which yields a payoff of 0. But player A can do strictly better by accepting $\pi - y > 0$! The Nash equilibrium is not subgame perfect.

Since this is an infinite horizon game, we cannot use backward induction to solve it. However, since every subgame that begins with an offer by some player is structurally identical with all subgames that begin with an offer by that player, we will look for an equilibrium with two intuitive properties:

1. no delay: whenever a player has to make an offer, the equilibrium offer is immediately accepted by the other player; and
2. stationarity: in equilibrium, a player always makes the same offer.

It is important to realize that at this point I do not claim that such equilibrium exists – we will look for one that has these properties. Also, I do not claim that if it does exist, it is the unique subgame perfect Nash equilibrium of the game. We will prove this later. However, given that the subgames are structurally identical, there is no a priori reason to think that offers must be non-stationary, and, if this is the case, that there should be any reason to delay agreement (given that doing so is costly). So it makes sense to look for a subgame perfect Nash equilibrium with these properties.
Let $x^*$ denote player A’s equilibrium offer and $y^*$ denote player B’s equilibrium offer (again, because of stationarity, there is only one such offer). Consider now some arbitrary time $t$ at which player A has to make an offer to player B. From the two properties, it follows that if B rejects the offer, he will then offer $y^*$ in the next period (stationarity), which A will accept (no delay). So, B’s payoff to rejecting A’s offer is $\delta y^*$. Subgame perfection requires that B reject any offer $\pi - x < \delta y^*$ and accept any offer $\pi - x > \delta y^*$. From the no delay property, this implies $\pi - x^* \geq \delta y^*$. However, it cannot be the case that $\pi - x^* > \delta y^*$ because player A could increase his payoff by offering some $x$ such that $\pi - x^* > \pi - x > \delta y^*$. Hence

$$\pi - x^* = \delta y^*$$

This equation states that in equilibrium, player B must be indifferent between accepting and rejecting player A’s equilibrium offer. By a symmetric argument it follows that in equilibrium, player A must be indifferent between accepting and rejecting player B’s equilibrium offer:

$$\pi - y^* = \delta x^*$$

The latter two equations have a unique solution:

$$x^* = y^* = \frac{\pi}{1 + \delta}$$

which means that there may be at most one subgame perfect Nash equilibrium satisfying the no delay and stationarity properties. The following proposition specifies this subgame perfect Nash equilibrium.

**Proposition 2.6.3** The following pair of strategies is a subgame perfect Nash equilibrium of the alternating-offers game:

- player A always offers $x^* = \frac{\pi}{1 + \delta}$ and always accepts offers $y \leq y^*$,
- player B always offers $y^* = \frac{\pi}{1 + \delta}$ and always accepts offers $x \leq x^*$.

**Proof.** We show that player A’s strategy as specified in the proposition is optimal given player B’s strategy. Consider an arbitrary period $t$ where player A has to make an offer. If he follows the equilibrium strategy, the payoff is $x^*$. If he deviates and offers $x < x^*$, player B would accept, leaving A strictly worse off. Therefore, such deviation is not profitable. If he instead deviates by offering $x > x^*$, then player B would reject. Since player B always rejects such offers and never offers more than $y^*$, the best that player A can hope for in this case is $\max\{\delta(\pi - y^*), \delta x^*\}$. That is, either he accepts player B’s offer in the next period or rejects it and A’s offer in the period after the next one is accepted. (Anything further down the road will be worse because of discounting.) However, $x^* > \delta x^*$ and also $\delta(\pi - y^*) = \delta x^* < x^*$, so such deviation is not profitable. Therefore, by the one-shot deviation principle, player A’s proposal rule is optimal given B’s strategy.

Consider now player A’s acceptance rule. At some arbitrary time $t$ player A must decide how to respond to an offer made by player B. From the above argument we know that player A’s optimal proposal is to offer $x^*$, which implies that it is optimal to accept an offer $y$ if and
only if $\pi - y > \delta x^*$. Solving this inequality yields $y < \pi - \delta x^*$ and substituting for $x^*$ yields $y \leq y'$, just as the proposition claims.

This establishes the optimality of player A’s strategy. By a symmetric argument, we can show the optimality of player B’s strategy. Given that these strategies are mutually best responses at any point in the game, they constitute a subgame perfect Nash equilibrium.

This is good but so far we have only proven that there is a unique subgame perfect Nash equilibrium that satisfies the no delay and stationarity properties. We have not shown that there are no other subgame perfect Nash equilibria in this game. The following proposition, whose proof involves knowing some (not much) real analysis, states this result.

**Proposition 2.6.4** The subgame perfect Nash equilibrium described in Proposition 1 is the unique subgame Nash perfect equilibrium of the alternating-offers game.

In equilibrium, the share depends on the discount factor and player A’s equilibrium share is strictly greater than player B’s equilibrium share. In this game there exists a “first-mover” advantage because player A is able to extract all the surplus from what B must forego if he rejects the initial proposal.

The Rubinstein bargaining model makes an important contribution to the study of negotiations. First, the stylized representation captures characteristics of most real-life negotiations: (a) players attempt to reach an agreement by making offers and counteroffers, and (b) bargaining imposes costs on both players.
3. STATIC GAMES OF INCOMPLETE INFORMATION

So far, we have only discussed games where players knew about each other’s utility functions. These games of complete information can be usefully viewed as rough approximations in a limited number of cases. Generally, players may not possess full information about their opponents. In particular, players may possess private information that others should take into account when forming expectations about how a player would behave.

3.1. Bayesian Games

To analyze these interesting situations, we begin with a class of games of incomplete information (i.e. games where at least one player is uncertain about another player’s payoff function) that are the analogue of the strategic form games with complete information: Bayesian games (static games of incomplete information). Although most interesting incomplete information games are dynamic (because these allow players to lie, signal, and learn about each other), the static formulation allows us to focus on several modeling issues that will come handy later.

Example 3.1.1 Consider the following simple example. There are two firms in some industry: an incumbent and a potential entrant. Incumbent decides whether to build a plant, and simultaneously Entrant decides whether to enter. Suppose that Entrant is uncertain whether Incumbent’s building cost is 1.5 or 0, while Incumbent knows his own cost. Entrant has only a belief that Incumbent has high cost with probability \( p \) (i.e., Incumbent has low cost with probability \( 1 - p \)). The payoffs are shown in Figure 3.1.

![Fig. 3.1 Entry Game with incomplete information](image)

Entrant’s payoff depends on whether Incumbent builds or not (but is not directly influenced by Incumbent’s cost). Entering for Entrant is profitable only if Incumbent does not build. Note that “don’t build” is a dominant strategy for Incumbent when his cost is high. However, Incumbent’s optimal strategy when his cost is low depends on his prediction about whether Entrant will enter. Denote the probability that player 2 enters with \( q \). Building is better than not building if

\[
1.5q + 3.5(1-q) \geq 2q + 3(1-q)
\]

or \( q \leq \frac{1}{7} \). In other words, a low-cost Incumbent will prefer to build if the probability that Entrant enters is less than \( \frac{1}{7} \). Thus, Incumbent has to predict Entrant’s action in order to choose his own action, while Entrant, in turn, has to take into account the fact that Incumbent will be conditioning his action on these expectations.

Definition 3.1.2 A Bayesian game \( G^B \) is a tuple
$$G^\text{B} = (I, \{A_i\}_i, \{\Theta_i\}_i, \{p_i\}_i, \{u_i\}_i)$$

where

$I = \{1,2,...,n\}$ is a set of players;

$A_i$ are player’s $i$, $i \in I$, action spaces;

$\Theta_i$ are player’s $i$, $i \in I$, type spaces;

$p_i : \Theta_i \rightarrow \Delta(\Theta_{-i})$ are player’s $i$, $i \in I$, beliefs about probability distribution over complementary type profiles which are common knowledge;

$u_i : A \times \Theta \rightarrow R$ are player’s $i$, $i \in I$, payoff functions

A Bayesian Game is finite, if, all $I$, $A_i$, and $\Theta_i$ for all $i \in I$ are finite.

Similarly to the games of complete information we create the Cartesian product of action spaces of all players, $A = A_1 \times A_2 \times ... \times A_n$, which we call an action profile space of the game. Analogously to this, we create the Cartesian product of type spaces of all players, $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$, which we call a type profile space of the game.

**Example 3.1.1 (Continued)** Now we rewrite the story as a Bayesian game. A set of players $I$ consists of two players:

$I = \{\text{Incumbent}, \text{Entrant}\}$

Their action spaces are

$A_i = \{\text{Build}, \text{Don't}\}$ and $A_E = \{\text{Enter}, \text{Don't}\}$

Incumbent can be either high-cost or low-cost. We will call these possibilities as Incumbent’s types. Entrant is only of one type, say $x$. This implies

$\Theta_i = \{\text{High-cost}, \text{Low-cost}\}$ and $\Theta_E = \{x\}$

Incumbent has beliefs that Entrant is of type $x$ with sure. Entrant has beliefs, that the probability of Incumbent being high-cost is $p$, while the probability of Incumbent being low-cost is $1 - p$, what we will write as

$p_i(x | \text{High-cost}) = 1, \quad p_i(x | \text{Low-cost}) = 1$

$p_E(\text{High-cost} | x) = p, \quad p_E(\text{Low-cost} | x) = 1 - p$

Profit functions are given in bi-matrices in Figure 3.1.

### 3.2. Strategic Form Representation of a Bayesian Game

Recall that the formal representation of a strategic form game of complete information was done by introducing the notation $G = (I, \{S_i\}_i, \{u_i\}_i)$, where $I = \{1,2,...,n\}$ is a set of players, $S_i$, are strategy spaces of all players, and $u_i$, $i \in I$, are payoff functions.

**Definition 3.2.1** In a Bayesian Game $G^\text{B}$, a pure strategy for player $i$ is a function which maps player $i$’s type into her action set.
Similarly, a mixed strategy is a type dependent probability distribution over $A_i$

$$\sigma_i : \Theta_i \rightarrow \Delta(A_i)$$

In other words, a strategy in a Bayesian game is a plan that specifies the action chosen by the player for his every type. This is a convenient way to specify strategies. It is as if players choose their type-contingent strategies before they learn their types, and then play according to these pre-committed strategies. It is similar, in some ways, to strategies in extensive form games that map information sets into actions at those information sets, and here the information sets are determined by nature’s choice of types.

This definition of strategies is very useful because it allows us to explicitly introduce the beliefs of players over strategies of their opponents when their opponents can be of different types, and each type can choose different actions.

**Example 3.2.2** In the game presented in Example 3.1.1, every Incumbent’s strategy consists of two actions: one of them is chosen when Incumbent is high-cost and the other is chosen when Incumbent is low-cost. Since Entrant is only of one type, his strategy coincides with his action.

$S_i = \{(Build, Build), (Build, Don't), (Don't, Build), (Don't, Don't)\}$ \hspace{1cm} and \hspace{1cm} $S_E = \{Enter, Don't\}$

For a long time, game theory was stuck because people could not figure out a way to solve such games. However, in a couple of papers in 1967-68, John C. Harsanyi proposed a method that allowed one to transform the game of incomplete information into a game of imperfect information, which could then be analyzed with standard techniques.

**Definition 3.2.3** (Harsanyi transformation) For a given Bayesian game $G^\theta$, we consider the following dynamic game of imperfect information:

1. Nature chooses a profile of types according to players’ beliefs.
2. Each player learns his own type and uses $p_i(\theta_j | \theta_i)$ to form beliefs over the other types.
3. Players simultaneously choose actions from $A_i$.
4. Payoffs $u_i(a | \theta_i) = u_i(a_1, a_2, ..., a_n | \theta_i)$ are realized for each player.

Note that in our setup we have $u_i$ depended on $\theta_i$ but not depend on $\theta_{-i}$. We call this setup one of private values, since each player’s payoff depends only on his private information and not on the private information of other players. We will only briefly mention the case of common values where payoffs are given by $u_i(a | \theta) = u_i(a_1, a_2, ..., a_n | \theta_1, \theta_2, ..., \theta_n)$, so that player $i$’s payoffs can indeed depend on the private information of the other players. This setup, called the common values case, has some interesting applications especially for auctions.

**Example 3.2.2 (Continued)** Harsanyi transformation applied to the Entry game leads to the dynamic game in Figure 3.2.
Fig. 3.2 The Harsanyi transformed Entry Game with incomplete information

Definition 3.2.4 An associated game to the Bayesian game $G^B = \{(I, \{A_i\}_{i \in I}, \{\Theta\}_{i \in I}, \{p_i\}_{i \in I}, \{u_i\}_{i \in I})\}$ is the strategic form game $G^* = (I, \{S_i\}_{i \in I}, \{v_i\}_{i \in I})$ where

- $I$ is the set of players in the game $G^B$;
- $S_i$, $i \in I$ are all players’ strategy spaces in the game $G^B$;
- $v_i : S \to R$, $i \in I$ are all players payoff functions of the form

$$v_i(s) = \sum_{\theta_i \in \Theta} p(\theta_i | \theta_j) u_i(a_i, s, \theta_j | \theta_j)$$

Similarly to the extensive form games, an associated game to two-player Bayesian game $G^B$ can be represented by a bi-matrix, in which each row corresponds to a strategy of the first player and each column corresponds to a strategy of the second player.

Example 3.2.2 (Continued) The associated game to Entry Game with incomplete information as presented in Figure 3.2 is as follows:

<table>
<thead>
<tr>
<th>Entrant</th>
<th>Incumbent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Build, Build</td>
<td>1.5-1.5p, -1</td>
</tr>
<tr>
<td>Build, Don’t</td>
<td>2-2p, 1</td>
</tr>
<tr>
<td>Don’t, Build</td>
<td>1.5+0.5p, 2p -1</td>
</tr>
<tr>
<td>Don’t, Don’t</td>
<td>2, 1</td>
</tr>
</tbody>
</table>

Fig. 3.3 The Harsanyi transformed Entry Game with incomplete information
3.3. **Bayesian Nash Equilibrium**

We are now ready to define a natural extension to the concept of Nash equilibrium that will apply to Bayesian games of incomplete information.

**Definition 3.3.1** The set of all Bayesian Nash equilibria of a Bayesian game coincides with the set of all Nash equilibria of the associated game to it.

**Proposition 3.3.2** A strategy profile $s^* \in S$, $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a pure strategy Bayesian Nash equilibrium of Bayesian game $G$ if and only if for all player $i$, $i \in I$, and every $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} u_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}) | \theta_i, \theta_{-i}) p_i(\theta_{-i} | \theta_i) \geq \sum_{\theta_{-i} \in \Theta_{-i}} u_i(s_i, s_{-i}^*(\theta_{-i}) | \theta_i, \theta_{-i}) p_i(\theta_{-i} | \theta_i)$$

for all $s_i \in S_i$, or equivalently

$$s_i^*(\theta_i) \in \arg \max_{s_i \in S_i} \sum_{\theta_{-i} \in \Theta_{-i}} u_i(s_i, s_{-i}^*(\theta_{-i}) | \theta_i, \theta_{-i}) p_i(\theta_{-i} | \theta_i)$$

for every player $i$, $i \in I$ and every $\theta_i \in \Theta_i$.

Simply stated, each type-contingent strategy is a best response to the type-contingent strategies of the other players. Player $i$ calculates the expected payoff of playing every possible type-contingent strategy $s_i(\theta_i)$ given his type $\theta_i$. To do this, he sums over all possible combinations of types for his opponents, $\theta_{-i}$, and for each combination, he calculates the expected payoff of playing against this particular set of opponents: The payoff $u_i(s_i, s_{-i}^*(\theta_{-i}) | \theta_i, \theta_{-i})$ is multiplied by the probability that this set of opponents $\theta_{-i}$ is selected by Nature: $p_i(\theta_{-i} | \theta_i)$. This yields the optimal behavior of player $i$ when of type $\theta_i$. We then repeat the process for all possible $\theta_i \in \Theta_i$ and all players.

A mixed strategy Bayesian Nash equilibrium can be defined in a similar way.

**Example 3.3.3** Now we find all Bayesian Nash equilibria for the Enter Game. To do this we find all Nash equilibria of the associated game given in Figure 3.3. First, we see that Incumbent’s strategy $(\text{Don't, Build})$ strictly dominates the strategy $(\text{Build, Build})$ as the strategy $(\text{Don't, Don't})$ does the strategy $(\text{Build, Don't})$. Eliminating the two strictly dominated strategies reduces the game to the one shown in Figure 3.4.

<table>
<thead>
<tr>
<th>Incumbent</th>
<th>Enter</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Don’t, Build</td>
<td>1.5+0.5$p$, 2$p$-1</td>
<td>3.5-0.5$p$, 0</td>
</tr>
<tr>
<td>Don’t, Don’t</td>
<td>2, 1</td>
<td>3, 0</td>
</tr>
</tbody>
</table>

**Fig. 3.3** The reduced strategic form of the game 3.3

If Entrant chooses Enter, then Incumbent’s unique best response is to choose $(\text{Don’t, Don’t})$ regardless of the value of $p < 1$. Hence $((\text{Don’t, Don’t}), \text{Enter})$ is a Nash equilibrium for all values of $p \in (0,1)$.

Note that Enter strictly dominates Don’t whenever $2p-1 > 0$, or $p > \frac{1}{2}$ and so Entrant will never mix in equilibrium in this case. Let’s then consider the cases when $p \leq \frac{1}{2}$. We now also have $((\text{Don't, Bild}), \text{Enter})$ as a Nash equilibrium. Suppose now that Entrant mixes in
equilibrium, i.e. he plays \textit{Enter} with probability $q$ and \textit{Don’t} with probability $1-q$. Since he is willing to randomize,

$$v_1((\text{Don’t, Bild}), \cdot) = v_1((\text{Don’t, Don’t}), \cdot)$$

or

$$q(1.5 + 0.5p) + (1 - q)(3.5 - 0.5p) = 2q + 3(1 - q)$$

or $q = \frac{1}{2}$. Similarly, suppose that Incumbent also mixes in equilibrium, i.e. he plays \textit{(Don’t, Bild)} with probability $r$ and \textit{(Don’t, Don’t)} with probability $1-r$. This is if

$$v_2(\cdot, \text{Enter}) = v_2(\cdot, \text{Don’t})$$

or

$$q(2p - 1) + (1 - q)$$

or $r = \frac{1}{2(1 - p)}$.

Hence, we have a mixed strategy Bayesian Nash equilibrium in which Incumbent plays \textit{(Don’t, Bild)} with probability $\frac{1}{2(1 - p)}$ and Entrant plays \textit{Enter} with probability $\frac{1}{2}$ for all $p \leq \frac{1}{2}$.

If $p = \frac{1}{2}$, then Entrant will be indifferent between her two pure strategies if Incumbent chooses \textit{(Don’t, Bild)} for sure, so he can randomize. Suppose he mixes in equilibrium with probability $q$ of playing \textit{Enter}. Then Incumbent’s expected payoff from \textit{(Don’t, Bild)} would be $3.25 - 1.5q$. Incumbent’s expected payoff from \textit{(Don’t, Don’t)} is then $3 - q$. He would choose \textit{(Don’t, Bild)} whenever $3.25 - 1.5q \geq 3 - q$, that is, whenever $q \leq \frac{1}{2}$. Hence, there is a continuum of mixed strategy Nash equilibria when $p = \frac{1}{2}$: Incumbent chooses \textit{(Don’t, Bild)} and Entrant randomizes with $q \leq \frac{1}{2}$. However, since $p = \frac{1}{2}$ is such a knife-edge case, we would usually ignore it in the analysis.

Summarizing the results, we have the following Nash equilibria:

- Neither the high nor low cost types build, and Entrant enters;

- If $p \leq \frac{1}{2}$, there are two types of equilibria:

  - the high-cost type does not build, but the low-cost type does, and Entrant enters;
  - the high-cost type does not build, but the low-cost type builds with probability $\frac{1}{2(1 - p)}$,

    and Entrant enters with probability $\frac{1}{2}$.

- If $p = \frac{1}{2}$, there is a continuum of equilibria: the high cost player 1 does not build, but the low-cost does, and Entrant enters with probability less than $\frac{1}{2}$.

Intuitively, the results make sense. The high-cost type never builds, so deterring Entrant’s entry can only be done by the low-cost type’s threat to build. If Entrant is expected to enter
for sure, then even the low-cost type would prefer not to build, which in turn rationalizes her decision to enter with certainty. This result is independent of her prior beliefs.
4. DYNAMIC GAMES OF INCOMPLETE INFORMATION

So far we have analyzed games in strategic form with and without incomplete information, and extensive form games with complete information. In this section we will analyze extensive form games with incomplete information. Many interesting strategic interactions can be modeled in this form, such as signaling games, repeated games with incomplete information in which reputation building becomes a concern, bargaining games with incomplete information, etc.

4.1. Bayes Conditions

The analysis of extensive form games with incomplete information will show that we need to provide further refinements of the Nash equilibrium concept. In particular, we will see that subgame perfect equilibrium concept that we have introduced when we studied extensive form games with complete information is not adequate.

Example 4.1.1 To illustrate the main problem in the subgame perfect equilibrium concept, however, the following game with imperfect, but complete, information is sufficient.

The strategic form of this game is given by

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Out</td>
<td>1, 3</td>
</tr>
<tr>
<td>Top</td>
<td>2, 1</td>
</tr>
<tr>
<td>Bottom</td>
<td>0, 2</td>
</tr>
</tbody>
</table>

Fig. 4.1 An illustrating game

Fig. 1.1 Strategic form of the game in Figure 4.1
It can be easily seen that the set of Nash equilibria of this game is \((\{\text{Top, Left}\}, \{\text{Out, Right}\})\). Since this game has only one subgame, i.e., the game itself, this is also the set of subgame perfect equilibria. But there is something implausible about the \((\text{Out, Right})\) equilibrium. Action \text{Right} is strictly dominated for player 2 at the information set \(I\). Therefore, if the game ever reaches that information set, player 2 should never play \text{Right}. Knowing that, then, player 1 should play \text{Top}, as he would know that player 2 would play \text{Left}; and he would get a payoff of 2 which is bigger than the payoff that he gets by playing \text{Out}. Subgame perfect equilibrium cannot capture this, because it does not test rationality of player 2 at the non-singleton information set \(I\).

The above discussion suggests the direction in which we have to strengthen the subgame perfect equilibrium concept. We would like players to be rational not only in every subgame but also in every continuation game.

**Definition 4.1.2** A continuation game \(\Gamma_h\) of an extensive form game \(\Gamma\) is a part of the game tree such that

4. it starts at an information set \(h\);
5. it contains every successors of all such \(x\) that \(x \in h\);
6. if it contains a node in an information set, then it contains all nodes in that information set.

A continuation game in the above example is composed of the information set \(I\) and the nodes that follow from that information set. First, notice that the continuation game does not start with a single decision node, and hence it is not a subgame. However, rationality of player 2 requires that he plays action \text{Left} if the game ever reaches there.

In general, the optimal action at an information set may depend on which node in the information set the play has reached. Consider the following modification of the above game.

![Fig. 4.3 A modification of the illustrating game](image)

Here the optimal action of player 2 at the information set \(I\) depends on whether player 1 has played \text{Top} or \text{Bottom} - information that 2 does not have. Therefore, analyzing player 2’s decision problem at that information set requires him forming beliefs regarding which decision node he is at. In other words, we require that
**Condition 4.1.3** (Bayes Condition 1: Beliefs) At each information set the player who moves at that information set has beliefs over the set of nodes in that information set.

and

**Condition 4.1.4** (Bayes Condition 2: Sequential Rationality) At each information set, strategies must be optimal, given the beliefs and subsequent strategies.

and

**Condition 4.1.5** (Bayes Condition 3: Weak Consistency) Beliefs are determined by Bayes’ Rule and strategies whenever possible.

Let us check what former two conditions imply in the game given in Figure 4.1. Bayes Condition 1 requires that player 2 assigns beliefs to the two decision nodes at the information set $I$: Let the probability assigned to the node that follows $Top$ be $\mu \in [0,1]$ and the one assigned to the node that follows $Bottom$ be $1 - \mu$. Given these beliefs the expected payoff to action $Left$ is

$$\mu \cdot 1 + (1 - \mu) \cdot 2 = 2 - \mu$$

whereas the expected payoff to $Right$ is

$$\mu \cdot 0 + (1 - \mu) \cdot 1 = 1 - \mu$$

Notice that $2 - \mu > 1 - \mu$ for any $\mu \in [0,1]$. Therefore, Bayes Condition 2 requires that in equilibrium player 2 never plays $Right$ with positive probability. This eliminates the subgame perfect equilibrium $(Out, Right)$, which, we argued, was implausible.

Although it requires players to form beliefs at non-singleton information sets, Bayes condition 1, does not specify how these beliefs are formed. As we are after an equilibrium concept, we require the beliefs to be consistent with the players’ strategies. As an example consider the game given in Figure 4.1 again. Suppose player 1 plays actions $Out$, $Top$, and $Bottom$ with probabilities $\alpha_1(Out)$, $\alpha_1(Top)$, and $\alpha_1(Bottom)$, respectively. Also let $\mu$ be the belief assigned to node that follows $Top$ in the information set $I$. If, for example, $\alpha_1(Top) = 1$ and $\mu = 0$; then we have a clear inconsistency between player 1’s strategy and player 2’s beliefs. The only consistent belief in this case would be $\mu = 1$. In general, we may apply Bayes’ Rule, whenever possible, to achieve consistency:

$$\mu = \frac{\alpha_1(Top)}{\alpha_1(Top) + \alpha_1(Bottom)}$$

Of course, this requires that $\alpha_1(Top) + \alpha_1(Bottom) \neq 0$, i.e., player 1 plays action $Out$ with probability 1, then player 2 does not obtain any information regarding which one of his decision nodes has been reached from the fact that the play has reached $I$.

### 4.2. Weak Perfect Bayesian Equilibrium

To be able to define the weak perfect Bayesian equilibrium more formally, let $H_i$ be the set of all information sets a player has in the game, and let $A(h)$ be the set of actions available at information set $h$. Recall from the Definition 2.1.9 that a behavioral strategy for player $i$,
in a extensive form game $\Gamma$ is a set of probability distributions over sets of feasible actions at all his information sets. Now, we redefine it more formally:

**Definition 4.2.1** A behavioral strategy for player $i \in I$, in a extensive form game $\Gamma$, is a function $\beta_i$, which assigns to each information set $h \in H_i$ a probability distribution on $A(h)$, i.e.,

$$\sum_{a \in A(h)} \beta_i(a) = 1$$

Let $B_i$ be the set of all behavioral strategies available for player $i$ and $B$ be the set of all behavioral strategy profiles, i.e., $B = B_1 \times B_2 \times \ldots \times B_n$.

**Definition 4.2.2** A belief system $\mu : X \rightarrow [0,1]$ assigns to each decision node $x$ in the information set $h$ a probability $\mu(x)$, where

$$\sum_{x \in h} \mu(x) = 1$$

for all $h \in H$.

Let $M$ be the set of all belief systems.

**Definition 4.2.3** A belief system combined with a behavioral strategy profile in an extensive form game $\Gamma$ with incomplete information $(\mu, \beta) \in M \times B$ is called an assessment.

Now we can define the concept of weak perfect Bayesian equilibrium.

**Definition 4.2.4** An assessment $(\mu, \beta)$ in an extensive form game $\Gamma$ with incomplete information that satisfies Bayes Condition 1 – 3 is a weak perfect Bayesian equilibrium of game $\Gamma$.

**Example 4.2.5** Consider the game in Figure 4.3. Let $\beta_i(a)$ be the probability assigned to action $a$ by player $i$ and $\mu$ be the belief assigned to the node that follows Top in information set $I$ Then an assessment in this game takes the form:

$$(\mu, \beta) = (\mu, (\beta_1, \beta_2), \mu, ((\beta_1(Out), \beta_1(Top), \beta_1(Bottom)), (\beta_2(Left), \beta_2(Right))))$$

In any weak perfect Bayesian equilibrium of this game we have $\beta_2(Left) = 1$ or $\beta_2(Left) = 0$, or $\beta_2(Left) \in (0,1)$.

Let us check each of the possibilities in turn:

(i) $\beta_2(Left) = 1$: In this case, sequential rationality of player 2 implies that the expected payoff to Left is greater than or equal to the expected payoff to Right; i.e.,

$$\mu \cdot 1 + (1 - \mu) \cdot 0 \geq \mu \cdot 0 + (1 - \mu) \cdot 2$$

or $\mu \geq \frac{1}{3}$. Sequential rationality of player 1 on the other hand implies that he plays Top; i.e., $\beta_1(Top) = 1$. Bayes’ rule then implies that

$$\mu = \frac{\beta_1(Top)}{\beta_1(Top) + \beta_1(Bottom)} = \frac{1}{1 + 0} = 1$$

which is greater than $\frac{1}{3}$, and hence does not contradict player 2’s sequential rationality.

Therefore, the following assessment is a weak perfect Bayesian equilibrium:
\[(\mu, \beta) = (1, ((0, 1, 0), (1, 0)))\]

(ii) \(\beta_2(\text{Left}) = 0\): Sequential rationality of player 2 now implies that \(\mu \leq \frac{1}{2}\), and sequential rationality of player 1 implies that \(\beta_1(\text{Out}) = 0\). Since \(\beta_1(\text{Top}) + \beta_1(\text{Bottom}) = 0\), however, we cannot apply Bayes’ rule, and hence Bayes Condition 3 is trivially satisfied. Therefore, there is a continuum of equilibria of the form

\[(\mu, \beta) = (\mu, ((0, 0, 0), (0, 1))), \mu \leq \frac{1}{2}\]

(iii) \(\beta_2(\text{Left}) \in (0, 1)\): Sequential rationality of player 2 implies that \(\mu = \frac{1}{2}\). For player 1 the expected payoff to \(\text{Out}\) is 1; to \(\text{Top}\) is \(2\beta_2(\text{Left})\), and to \(\text{Bottom}\), is 0: Clearly, player 1 will never play \(\text{Bottom}\) with positive probability, that is in this case we always have \(\beta_1(\text{Bottom}) = 0\). If \(\beta_1(\text{Out}) = 1\), then we must have \(2\beta_2(\text{Left}) \leq \frac{1}{2}\), and we cannot apply Bayes’ rule. Therefore, any assessment that has

\[(\mu, \beta) = \left(\frac{1}{2}, ((1, 0, 0), (q, 1-q))\right), \quad q \leq \frac{1}{2}\]

is a weak perfect Bayesian equilibrium. If, on the other hand, \(\beta_1(\text{Out}) = 0\), then we must have \(\beta_1(\text{Top}) = 1\), and Bayes’ rule implies that \(\mu = 1\), contradicting \(\mu = \frac{1}{2}\). If \(\beta_1(\text{Out}) \in (0, 1)\), then Bayes’ rule implies that \(\mu = 1\) again contradicting \(\mu = \frac{1}{2}\).

4.3. Sequential Equilibrium

Weak perfect Bayesian equilibrium could be considered a weak equilibrium concept, because it does not put enough restrictions on out-of-equilibrium beliefs. The most commonly used equilibrium concept that do not suffer from such deficiencies is that of sequential equilibrium. Before we can define sequential equilibrium, however, we have to define a particular consistency notion.

**Definition 4.3.1** A behavioral strategy profile \(\beta \in \mathcal{B}\) in an extensive form game \(\Gamma\) of incomplete information is said to be completely mixed if every action receives positive probability.

**Definition 4.3.2** An assessment \((\mu, \beta)\) in an extensive form game \(\Gamma\) with incomplete information is consistent if there exists a completely mixed sequence \((\mu^n, \beta^n)\) that converges to \((\mu, \beta)\) such that \(\mu^n\) is derived from \(\beta^n\) using Bayes’ rule for all \(n\).

**Definition 4.3.3** An assessment \((\mu, \beta)\) in an extensive form game \(\Gamma\) with incomplete information is sequential equilibrium of the game if it is sequentially rational and consistent.
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