The study of continuous selections begins with the paper of Michael [37] and it is an active area of research in general topology. Michael proved that any lower semicontinuous multifunction $F : X \rightarrow Y$ with closed convex values admits a continuous selection, where $X$ is a paracompact topological space and $Y$ is a Banach space.

For continuous multifunctions which are defined on the segment $[0, 1]$ with compact values in Euclidean space $E^n$ the existence of a continuous selection was proved by Filippov [12] for multifunctions satisfying the Lipschitz condition. Hermes [17] proved existence of a continuous selection for multifunctions with bounded variation. Several authors obtained a continuous selection theorem for a multifunction with non convex values in a Banach space of functions (e.g. Fryszkowski [14]).

### 7.1 Convex-valued Selection Theorem

This section deals with the convex-valued selection theorem. For the first time it was proved by Michael. What we give here is a version published in [45].

**Paracompactness of the domain as a necessary condition**

**THEOREM 7.1.** Let $X$ be a topological space such that each lower semicontinuous map from $X$ into any Banach space with closed convex values admits a continuous single-valued selection. Then every open covering of $X$ admits a locally finite open refinement.

**Proof.** The proof is divided into two parts:

1. **Construction:**

Let
(1) \( \gamma = \{G_{\alpha}\}_{\alpha \in A} \) be an open covering of the space \( X \); 

(2) \( B = l_1(A) \) be the Banach space of all summable functions \( s : A \rightarrow \mathbb{R} \) over the index set \( A \); and 

(3) For every \( x \in X \), let 
\[
F(x) = \{s \in B \mid s \geq 0, \|s\| = 1 \text{ and } s(\alpha) = 0, \text{ whenever } x \not\in G_{\alpha}\}.
\]

We claim that then:

(a) \( F(x) \) is a nonempty convex closed subset of the Banach space \( B \), for every \( x \in X \); and 

(b) The map \( F : X \rightarrow B \) is lower semicontinuous, i.e. for every \( x \in X \), every \( s \in F(x) \) and every \( \epsilon > 0 \), the pre-image \( F^{-1}(D(s, \epsilon)) \) contains an open neighbourhood of \( x \).

It follows from the hypothesis of the theorem that there exists a continuous selection for \( F \), say \( f \). Let:

(4) \( e_{\alpha} = [f(x)](\alpha) \); 

(5) \( e(x) = \sup\{e_{\alpha}(x) \mid \alpha \in A\} \); and 

(6) \( V_{\alpha} = \{x \in X \mid e_{\alpha}(x) > \frac{e(x)}{2}\} \).

We claim that then:

(d) \( e \) is a continuous positive function; 

(e) If \( e_{\alpha}(x) > 0 \) then \( x \in G_{\alpha} \) for all \( \alpha \in A \); 

(f) \( V_{\alpha} \subset G_{\alpha} \), for all \( \alpha \in A \); 

(g) \( \{V_{\alpha}\} \) is a locally finite family of open subsets of the space \( X \); and 

(h) The family \( \{V_{\alpha}\}_{\alpha \in A} \), is a cover of the space \( X \).

II. Verification:

(a) Let \( A(x) = \{\alpha \in A \mid x \in G_{\alpha}\} \). \( F(x) \) is the standard basic simplex in the Banach space \( l_1(A(x)) \),
(b) For \( x \in X, s \in F(x) \) and \( \epsilon > 0 \), let us first consider the case when
\[
\text{supp}(s) = \{ \alpha \in A \mid s(\alpha) > 0 \} = \{ \alpha_1, \alpha_2, \ldots, \alpha_N \}
\]
is a finite subset of \( A \). Then due to the construction of the mapping \( F \), the point \( s \) belongs to \( F(x') \), for every \( x' \) from the neighbourhood \( G(x) = \cap_{i \in \mathbb{N}} G_{\alpha_i} \) of the point \( x \). Hence
\[
G(x) \subset F^{-1}(\{s\}) \subset F^{-1}(D(s, \epsilon)),
\]
i.e. \( F \) is lower semicontinuous at \( x \). The second case of countable \( \text{supp}(s) \) follows from the first case and from the obvious fact that in the standard simplex of the space \( l_1 \) the subset of points with finite supports constitutes a dense subset.

(c) The function \( e_\alpha : X \to [0, 1] \) is a composition of the continuous selection \( f \) and the \( \alpha \)-th coordinate projection \( p_\alpha \) of the entire Banach space \( l_1(A) \). The equality \( \sum_{\alpha \in A} e_\alpha(x) = 1 \) follows from (4) and therefore \( f(x) \in F(x) \).

(d) For an arbitrary \( x \in X \), we pick an index \( \beta = \beta(x) \) such that \( e_\beta(x) > 0 \). Then for some finite set of indices \( \Gamma(x) \subset A \) we have that
\[
1 - \sum_{\alpha \in \Gamma(x)} e_\alpha(x) < \frac{e_\beta(x)}{2}.
\]
On the left side is the sum of a finite number of continuous functions. Hence, the inequality
\[
\sum_{\alpha \notin \Gamma(x)} e_\alpha(z) = 1 - \sum_{\alpha \in \Gamma(x)} e_\alpha(z) < \frac{e_\beta(z)}{2}
\]
holds for every \( z \) from some open neighbourhood \( W(x) \) of the point \( x \). But then \( e_\gamma(x) < e_\beta(z) \), for all \( \gamma \notin \Gamma(x) \). So we have proved that the function \( e(.) \) is in fact the maximum of a finite number of continuous functions in the neighbourhood \( W(x) \). Therefore \( e(.) \) is continuous. Finally, positivity of \( e(.) \) follows from (c).

(e) If \( x \notin G_\alpha \) then for every \( s \in F(x) \), we have that \( s(\alpha) = 0 \), (see (3)). So by \( f(x) \in F(x) \), we get \( e_\alpha(x) = |f(x)|(\alpha) = 0 \).

(f) This follows from
\[
e_\alpha(x) > \frac{e(x)}{2} \geq \frac{e_\beta(x)}{2},
\]
from (e), and from \( e_\beta(x) > 0 \) (see the proof of (d)).

(g) It follows from (6) and by continuity of functions \( e_\alpha \) and \( e \) that \( V_\alpha \) is an open set. As in the proof of (d) we find for an arbitrary \( x \), some finite set \( \Gamma(x) \subset A \) and some neighbourhood
Then the mapping \( \epsilon > \) some semicontinuous mapping. Let \( f \) II. Verification

We claim that then:

I. Construction

\[
\epsilon > \frac{e(z)}{2} > 1 - \sum_{\alpha \in \Gamma(x)} e_\alpha(z) = \sum_{\alpha \in \Gamma(x)} e_\alpha(z).
\]

Hence \( e_\gamma(z) > e_\alpha(z) \), for every \( \alpha \in \Gamma(x) \), i.e. \( \gamma \in \Gamma(x) \).

(h) It is followed by the contradiction: if \( x \notin \bigcup V_\alpha \), \( \alpha \in A \), then \( e_\alpha(x) \leq \frac{e(x)}{2} \) and

\[
0 < e(x) = \text{supp}(e_\alpha(x) \mid \alpha \in A) \leq \frac{e(x)}{2}.
\]

In what follows we will need the following two propositions:

**Proposition 7.2.** Let \( X \) be a topological space, \( Y \) be a metric space. Let \( F : X \rightarrow (Y, \rho) \) be a lower semicontinuous mapping. Let \( f : X \rightarrow Y \) be a single-valued continuous mapping such that for some \( \epsilon > 0 \), the intersection of \( F(x) \) with the open \( \epsilon \)-balls \( D(f(x), \epsilon) \) are nonempty, for all \( x \in X \). Then the mapping \( G : X \rightarrow Y \), defined by \( G(x) = F(x) \cap D(f(x), \epsilon) \), is lower semicontinuous.

**Proof.** In two steps:

I. Construction: Let

1. \( U \) be open in \( Y \) and \( G^{-1}(U) \) nonempty;
2. \( x \in G^{-1}(U) \) and \( y \in G(x) \cap U = F(x) \cap D(f(x), \epsilon) \cap U; \)
3. \( \epsilon_1 > 0 \) be such that the closed ball \( \text{Cl}(y, \epsilon_1) \) is contained in the open set \( D(f(x), \epsilon) \cap U; \) and
4. \( D(f(x), \delta) \) be a small open ball centered at \( f(x) \); more precisely, let \( 0 < \delta < \epsilon - (\epsilon_1 + \rho(f(x), y)). \)

We claim that then:

a. If \( z \in D(f(x), \delta) \) then \( \text{Cl}(y, \epsilon_1) \subset D(z, \epsilon); \)

b. \( f^{-1}(D(f(x), \delta)) \cap F^{-1}(D(y, \epsilon_1)) \subset G^{-1}(U); \)

c. The intersection from (b) is a nonempty open neighbourhood of \( x; \) and

(d) \( G^{-1}(U) \) is open in \( X \).

II. Verification:

a. Clearly, for every \( y' \in Y \), \( \rho(y', z) \leq \rho(y', y) + \rho(y, f(x)) + \rho(f(x), z). \) So, if \( \rho(y', y) \leq \epsilon_1 \) and \( \rho(f(x), z) < \delta < \epsilon - \epsilon_1 - \rho(f(x), y) \)

then \( \rho(y', z) < \epsilon. \)
(b) If \( x' \in f^{-1}(D(f(x), \delta)) \) then the point \( z = f(x') \) lies in the open ball \( D(f(x), \delta) \) and (see (a)) \( D(y, \varepsilon_1) \subset D(z, \varepsilon) \). If, in addition, \( x' \in F^{-1}(D(y, \varepsilon_1)) \) then the set \( F(x') \) intersects the ball \( D(y, \varepsilon_1) \subset U \). So there exists \( y' \):
\[
y' \in F(x') \cap D(y, \varepsilon_1) \subset F(x') \cap D(f(x'), \varepsilon) \cap U
\]
i.e. \( x' \in G^{-1}(U) \).

(c) It follows from the continuity of \( f \) and lower semicontinuity of \( F \) at the point \( x \).

(d) It follows from (b) and (c) because \( x \) is an arbitrary point of \( G^{-1}(U) \).

□

Proposition 7.3. Let \( \{e_\alpha\}_{\alpha \in A} \) be a locally finite partition of unity on a topological space \( X \) and let \( \{y_\alpha\}_{\alpha \in A} \) be arbitrary points from a topological vector space \( Y \). Then the map \( f : X \to Y \) defined by
\[
f(x) = \sum_{\alpha \in A} e_\alpha(x)y_\alpha
\]
is continuous.

Proof. It suffices to observe that for a fixed \( x \in X \), the mapping \( f \) is a sum of a finite number of continuous mappings \( f_\alpha(x) = e_\alpha(x)y_\alpha \) in some suitable neighbourhood of this point. □

Here is the Repovs-Semenov’s proof of the convex-valued Selection Theorem:

**THEOREM 7.4.** Let \( X \) be a paracompact space, \( B \) a Banach space and \( F : X \rightharpoonup B \) a lower semicontinuous map with nonempty closed convex values. Then \( F \) admits a continuous single-valued selection.

We obtain theorem 7.4 as a corollary of the following two propositions. The first one establishes the existence of some \( \varepsilon \)-selection. The second one provides the existence of a uniformly convergent sequence \( \{f_n\} \) of \( \varepsilon_n \)-selections of the given multi-valued mapping.

**Definition 7.5.** Let \( F : X \rightharpoonup Y \) be a multi-valued mapping of a topological space \( X \) into a metric space \( (Y, \rho) \). Then a single-valued mapping \( f : X \to Y \) is said to be an \( \varepsilon \)-selection of \( F \) if
\[
dist(f(x), F(x)) < \varepsilon,
\]
for all \( x \in X \), where
\[
dist(f(x), F(x)) = \inf\{\rho(f(x), y) \mid y \in F(x)\}.
\]

The fact that \( f \) is an \( \varepsilon \)-selection of \( F \) geometrically means that every open ball \( D(f(x), \varepsilon) \) intersects the set \( F(x) \), for every \( x \in X \).
Proposition 7.6. Let $X$ be a paracompact space, $B$ a normed space and $F : X \rightharpoonup B$ a convex-valued lower semicontinuous map. Then for every $\epsilon > 0$ exists a continuous single-valued $\epsilon$-selection $f_\epsilon : X \to B$ of the map $F$.

Proposition 7.7. Let $X$ be a paracompact space, $B$ a normed space and $F : X \rightharpoonup B$ a convex-valued lower semicontinuous map. Then for every sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ of positive numbers, converging to zero, there exists a uniformly Cauchy sequence $\{f_n\}$ of continuous single-valued $\epsilon_n$-selections $f_n : X \to B$ of the map $F$.

Proof of Theorem 7.4. Choose a converging sequence $\epsilon_n \to 0$, $\epsilon_n > 0$ and let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence of continuous single-valued $\epsilon_n$-selections $f_n : X \to B$ of the map $F$ constructed in Proposition 7.7.

Pick $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\epsilon_n < \epsilon/3$ and $\|f_n(x) - f_{n+p}(x)\| < \epsilon/3$, for all $n > N$, $p \in \mathbb{N}$, and $x \in X$. For each $x \in X$ and for each $n \in \mathbb{N}$, we can find an element $z_n(x) \in F(x)$ such that

$$\|z_n(x) - f_n(x)\| < \epsilon_n.$$

Hence

$$\|z_n(x) - z_{n+p}(x)\| \leq \|z_n(x) - f_n(x)\| + \|f_n(x) - f_{n+p}(x)\| + \|f_{n+p}(x) - z_{n+p}(x)\| < \epsilon_n + \frac{\epsilon}{3} + \epsilon_{n+p} < \epsilon.$$

Therefore $\{z_n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete subspace $F(x) \subset B$ of the metric space $B$ and there exists

$$\lim_{n \to \infty} z_n(x) = z(x) \in F(x).$$

Finally, the equality

$$\lim_{n \to \infty} \|z_n(x) - f_n(x)\| = 0$$

implies that there exists

$$\lim_{n \to \infty} f_n(x) = f(x),$$

and that $z(x) = f(x)$. Hence $f(x) \in F(x)$ and the map $f$ is continuous as the point-wise limit of a uniformly Cauchy sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous functions. \hfill \Box

Proof of proposition 7.6. In two steps:

I. Construction: For a given $\epsilon > 0$ and for every $y \in B$ let:

1. $D(y, \epsilon) = \{z \in B \mid \|z - y\| < \epsilon\}$ be an open ball in $B$ with the radius $\epsilon$ centered at $y$; and

2. $U(y, \epsilon) = F^{-1}(D(y, \epsilon)) = \{x \in X \mid F(x) \cap D(y, \epsilon) \neq \emptyset\}$.

We claim that then:

(a) $\{U(y, \epsilon)\}_{y \in B}$, is an open covering of the space $X$; and
(b) There exists a locally finite partition of unity \(\{e_\alpha\}_{\alpha \in A}\) inscribed into the covering \(\{U(y, \epsilon)\}_{y \in B}\). Let:

(3) \(y_\alpha\) be an arbitrary element of \(B\) such that \(\text{supp}(e_\alpha) \subset U(y_\alpha, \epsilon)\); and

(4) Let \(f_\epsilon(x) = \sum_{\alpha \in A} e_\alpha(x)y_\alpha\).

We claim that then:

(c) \(f_\epsilon\) is a well-defined continuous mapping; and

(d) \(\text{dist}(f_\epsilon(x), F(x)) < \epsilon\), for all \(x \in X\).

II. Verification:

(a) It follows from the definition of the lower semicontinuity of the map \(F\);

(b) It follows from the paracompactness of the space \(X\);

(c) It follows from Proposition 7.3;

(d) For a given \(x \in X\), let

\[
\{\alpha \in A \mid x \in \text{supp}(e_\alpha)\} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}.
\]

Then \(x \in \text{supp}(e_\alpha) \subset U(y_\alpha, \epsilon)\), i.e. \(F(x) \cap F(y_\alpha, \epsilon) \neq \emptyset\). Hence \(\|z_i - y_\alpha\| < \epsilon\), for some \(z_i \in F(x); i \in \{1, 2, \ldots, n\}\). Let \(z = \sum_{i=1}^n e_\alpha(x)z_i\). By the convexity of the set \(F(x)\) we have \(z \in F(x)\) and by the convexity of open balls in a normed space we have

\[
\text{dist}(f_\epsilon(x), F(x)) \leq \|f_\epsilon(x) - z\| \leq \left\| \sum_{i=1}^n e_\alpha(x)(y_\alpha - z_i) \right\| \leq \sum_{i=1}^n e_\alpha(x)||y_\alpha - z_i|| \leq \epsilon \sum_{i=1}^n e_\alpha(x) = \epsilon.
\]

\(\square\)

Proof of proposition 7.7. In two steps:

I. Construction: We shall construct by induction a sequence of convex-valued lower semicontinuous mappings \(\{F_n : X \rightarrow B\}_{n \in \mathbb{N}}\) such that:

(i) \(F(x) = F_0(x) \supset F_1(x) \supset \cdots \supset F_n(x) \supset F_{n+1}(x) \supset \ldots\), for all \(x \in X\);

(ii) \(\text{diam}F_n(x) \leq 2\epsilon_n\); and
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(iii) \( f_n \) is and \( \varepsilon_n \)-selection of the mapping \( F_{n-1} \), for every \( n \in \{1, 2, \ldots \} \).

Base of induction: We apply Proposition 7.6 for the spaces \( X \) and \( B \), the mapping \( F = F_0 \), and for the number \( \varepsilon = \varepsilon_1 \). In such a way we find a continuous \( \varepsilon_1 \)-selection \( f_1 \) of the map \( F_0 \). Let

\[
F_1(x) = F_0(x) \cap D(f_1(x), \varepsilon_1),
\]

where \( D(f_1(x), \varepsilon) \) is the open ball in \( B \) of radius \( \varepsilon_1 \), centered at the point \( f_1(x) \). We claim that then:

(a1) \( F_1(x) \) is a nonempty convex subset of \( F_0(x) \);

(b1) \( \text{diam} F_1(x) \leq 2\varepsilon_1 \); and

(c1) The mapping \( F_1 : X \rightharpoonup B \) is lower semicontinuous.

Inductive step: Suppose that the mappings \( F_1, F_2, \ldots, F_{m-1}, f_1, f_2, \ldots, f_{m-1} \) with the properties (i)-(iii) have already been constructed. We apply Proposition 7.6 for spaces \( X \) and \( B \), mapping \( F_{m-1} \) and for the number \( \varepsilon_m > 0 \) and find a continuous \( \varepsilon_m \)-selection \( f_m \) of the map \( F_{m-1} \). Let

\[
F_m(x) = F_{m-1}(x) \cap D(f_m(x), \varepsilon_m).
\]

We claim that then:

(am) \( F_m(x) \) is a nonempty convex subset of \( F_{m-1} \);

(bm) \( \text{diam} F_m(x) \leq 2\varepsilon_m \); and

(cm) The mapping \( F_m : X \rightharpoonup B \) is lower semicontinuous.

Next, we claim that then:

(d) The sequence \( \{f_n\}_{n \in \mathbb{N}} \) is a uniformly Cauchy sequence of continuous single-valued \( \varepsilon_n \)-selections \( f_n : X \to B \) of the map \( F \).

II. Verification:

(a1) Follows since \( f_1 \) is an \( \varepsilon_1 \)-selection of \( F_0 \) and because the intersection of convex sets is again a convex set;

(b1) Follows since \( F_1(x) \) is a subset of a ball of radius \( \varepsilon_1 \); and

(c1) It follows from Proposition 7.2.

(am)-(cm) can be proved similarly as (a1)-(c1).
(d) $f_n$ is a continuous $\epsilon_n$-selection of the mapping $F_{n-1}$ and $F_{n-1}(x) \subset F(x)$. Hence $f_n$ is a continuous $\epsilon_n$-selection of $F$. From the inclusion $F_{n+p}(x) \subset F_n(x)$ and by condition (ii) we have that for every $n, p \in \mathbb{N}$ and $x \in X$,

$$
\|f_n(x) - f_{n+p}(x)\| \leq \text{dist}(f_n(x), F_n(x)) + \text{diam}F_n(x) + \text{dist}(f_{n+p}(x), F_{n+p}(x)) < 3\epsilon_n + \epsilon_{n+p}.
$$

Since $\epsilon_n \to 0$ we thus obtain (d).

□

The convex-valued theorem has many versions and generalizations. We cite here two of them:

**THEOREM 7.8.** Let $X$ be a paracompact topological space, $(Y, d)$ be a locally convex metric vector space and $F : X \rightrightarrows Y$ a lower semicontinuous multifunction with complete convex values. Then $F$ has a continuous single-valued selection.

**THEOREM 7.9.** Let $X$ be a paracompact topological space, $(Y, d)$ a locally convex metric vector space and $F : X \rightrightarrows Y$ a lower semicontinuous multifunction with complete convex values. Then for every $\epsilon > 0$ and for every continuous single-valued $\epsilon$-selection $f_\epsilon$ of $F$, there exists a continuous single-valued selection $f$ of $F$ such that

$$
d(f_\epsilon(x), f(x)) \leq \epsilon, \quad \text{for all } x \in X.
$$

### 7.2 Continuous selections from topological to metric spaces

In the monograph *Selections theorems and their applications* (Lecture Notes Math. 263) T. Parthasarathy wrote:

*It could be nice if one could prove a selection theorem when $Y$ is an arbitrary metric space ($X$ being topological) - of course the assumptions on the map $F$ as well as the sets $F(x)$ should be stronger.*

As the following example shows, it is not obvious, which assumptions we should take. Even a nice multifunction with exactly two values at each $x$ does not have necessarily a continuous selection.

**Example 7.10.** Let $A = [0, 2], B = [0, 4]$ be closed real intervals. Let $X$ be a circle (a quotient space) obtained from $A$ by identifying the points 0 and 2. Let $Y$ be a circle obtained from $B$ by identifying the points 0 and 4. Let us denote the identified points 0 and 2 in $A$ by $s$ and 0 and 4 in
7.2 Continuous selections from topological to metric spaces

We now define $F: X \rightharpoonup Y$ as follows:

$$F(s) = \{2, z\}$$
$$F(t) = \{t, 2 + t\} \quad \text{for each } t \in (0, 2).$$

It is easy to see that $F$ is l.s.c. at each $t$ in $X$. $B$ is metrizable and according to Lemma 7.13 $F$ is Hausdorff continuous.

But there is no continuous selection $f: X \to Y$ for $F$. If a continuous function $g: X \to Y$ were a selection for $F$ then it would be defined by $g(t) = t$ for all $t \neq s$ or $g(t) = 2t$ for all $t \neq s$. Let us take the sequence $\{a_n\}_{n=1}^{\infty}$ defined by:

$$a_{2k} = 2 - \frac{1}{k}$$
$$a_{2k+1} = \frac{1}{k}, \quad k = 1, 2, 3, \ldots .$$

Then $\{a_n\}_{n=1}^{\infty}$ converges to $s$ but the sequence $\{g(a_n)\}_{n=1}^{\infty}$ has two limit points.

The following theorem gives an answer to the Parthasarathy’s question:

**Theorem 7.11.** Let $X$ be a topological space, let $(Y, d)$ be a metric space. Let $F: X \rightharpoonup Y$ be a Hausdorff continuous multifunction such that for each $x \in X$ the set $F(x)$ is not a singleton. Suppose that there exists a uniformly continuous function $v: Y \to \mathbb{R}$ such that

(i) For each $x$ in $X$ the set $v(F(x))$ is bounded below.

(ii) For each $x$ in $X$

$$\inf(|v(a) - v(b)|; \ a, b \in F(x) \text{ and } a \neq b) > 0 \quad \text{holds.}$$

Then $F$ has a continuous selection.

Before proving Theorem 7.11 we need the following technical lemma.

**Lemma 7.12.** Let $X$ be a topological space. Let $(Y, d)$ be a metric space. Let $F: X \rightharpoonup Y$ be a Hausdorff continuous multifunction such that for each $x \in X$ the set $F(x)$ is not a singleton. Let $v: Y \to \mathbb{R}$ be a uniformly continuous function. If we define a function $c: X \to Y$ as follows:

$$c(x) = \inf(|v(a) - v(b)|; \ a, b \in F(x) \text{ and } a \neq b)$$

then the function $c$ is continuous.
Proof. Let \( x \in X \). Let \( \epsilon > 0 \) be arbitrary. Using the uniform continuity of \( v \) we see that there exists \( r > 0 \) such that

\[
\forall a, b \in Y \text{ such that } d(a, b) < r \quad |v(a) - v(b)| < \frac{\epsilon}{4}
\]

holds.

Let \( U(x) \) be a neighbourhood of \( x \) such that

(i) for each \( t \) in \( U(x) \)

\[
H(F(t), F(x)) < r
\]

where \( H \) denotes the Hausdorff distance induced by \( d \).

Let \( t \in U(x) \).

(1) To prove \( c(t) < c(x) + \epsilon \) let \( z, s \in F(x) \) such that

(ii)

\[
||v(s) - v(z)|| - c(x) < \frac{\epsilon}{4}
\]

By (i) there exist \( z', s' \in F(t) \) such that

\[
d(z, z') < r \quad \text{and} \quad d(s, s') < r.
\]

Hence

(iii)

\[
||v(z') - v(s')|| - |v(z) - v(s)| < \frac{\epsilon}{2}
\]

holds.

Therefore by (ii) and (iii)

\[
|v(z') - v(s') - c(x)| < \epsilon, \quad \text{so} \quad |v(z') - v(s')| < c(x) + \epsilon
\]

and since \( c(t) < |v(z') - v(s')|, \) \( c(t) < c(x) + \epsilon \) holds.

(2) To prove \( c(x) < c(t) + \epsilon \) let \( p, q \) be elements of \( F(t) \) such that

\[
||v(p) - v(q)|| - c(t) < \frac{\epsilon}{4}
\]

By (i) there exist \( p', q' \) from \( F(x) \) such that

\[
d(p, p') < r \quad \text{and} \quad d(q, q') < r
\]

so

\[
|v(p) - v(p')| < \frac{\epsilon}{4}
\]

and

\[
|v(q) - v(q')| < \frac{\epsilon}{4}
\]

The rest of the proof of (2) is the same as in (1) and it is left to the reader. So we have \( \forall t \in U(x) : |c(x) - c(t)| < \epsilon \) and the proof is completed.

\[ \square \]
Proof of Theorem 7.11. First let us define a function \( c : X \to \mathbb{R} \) as follows:

\[
    c(x) = \inf |v(a) - v(b)|; \quad a, b \in F(x) \text{ and } a \neq b.
\]

By Lemma 7.12 \( c \) is continuous and by (ii) \( c(x) > 0 \) for each \( x \) in \( X \). It is easy to verify, using (i) and (ii) that for each \( x \in X \) the set \( v(F(x)) \) has the least element and that there exists exactly one point \( y \in F(x) \) such that \( v(y) = \min v(F(x)) \).

Let us denote the minimizing element \( y \) as \( f(x) \). Then \( f \) is a function from \( X \) into \( Y \). Obviously \( f \) is a selection for \( F \). We will prove that \( f \) is continuous.

To prove this let \( x \in X \) and \( \epsilon > 0 \). We will show that there is an open set \( U(x) \) in \( X \) such that \( x \in U(x) \) and for each \( t \in U(x) \) \( d(f(t), f(x)) < \epsilon \) holds.

First take \( r > 0 \) such that \( c(x) > r \).

Next let \( h > 0 \) be such that

\[
    h < \epsilon \quad \text{and for all } s, z \in Y \text{ such that } d(s, z) < h \text{ the inequality } |v(s) - v(z)| < r
\]

holds.

Since the function \( c \) is continuous at \( x \) and \( F \) is Hausdorff continuous at \( x \) there exists an open neighbourhood \( O(x) \) of \( x \) such that

(a) for each \( t \in O(x) \) \( c(t) > r \) and

(b) for each \( t \in O(x) \) \( H(F(x), F(t)) < h \) hold.

Suppose that, contrary to what we wish to prove,

(c) There exists \( s \in O(x) \) such that \( d(f(s), f(x)) \geq \epsilon \)

Using (b) and the fact that \( f \) is a selection for \( F \) we obtain

(b1) There exists \( m \in F(s) \) such that \( d(f(x), m) < h \)

(b2) There exists \( n \in F(x) \) such that \( d(f(s), n) < h \)

It follows from (c) that \( m \neq f(s) \) and \( n \neq f(x) \). Now using (v) it follows that

(v1) \[ |v(m) - v(f(x))| < r \]

(v2) \[ |v(n) - v(f(s))| < r. \]
7.2 Continuous selections from topological to metric spaces

It is clear now (using the definition of \( f \) and \( c \), (a) and the assumption (ii)) that

\[ v(f(s)) + r < v(m) \]

and (v1) implies

\[ v(f(s)) < v(f(x)). \]

Analogically \( v(f(x)) + r < v(n) \) holds and (v2) implies

\[ v(f(x)) < v(f(s)). \]

This is a contradiction. Hence for each \( s \in O(x) \): \( d(f(s), f(x)) < \epsilon \) holds and the proof is completed. \( \square \)

The previous results can be improved in the case of multifunctions with exactly \( n \) values.

**Lemma 7.13.** Let \( X \) be a topological space. Let \( Y \) be a metric space. Let \( n > 0 \) be an integer. Let \( F : X \rightarrow Y \) be a l.s.c. multifunction with exactly \( n \) values for each \( x \in X \). Then \( F \) is Hausdorff continuous.

**Proof.** Let \( a \) be an arbitrary point of \( X \). Let \( \epsilon > 0 \) be arbitrary. Let us denote \( V = B_\epsilon(F(a)) \) and let \( F(a) = \{a_1, a_2, \ldots, a_n\} \). Hence

\[ V = \bigcup_{i=1}^{n} B_\epsilon(\{a_i\}). \]

For \( i = 1, 2, \ldots, n \) there exists a neighbourhood \( U_i \) of \( a \) such that if \( t \) is in \( U_i \) then \( F(t) \cap B_\epsilon(\{a_i\}) \neq \emptyset \).

Denote \( U = \bigcap_{i=1}^{n} U_i \). Then if \( t \) is in \( U \)

\[ F(t) \cap B_\epsilon(\{a_i\}) \neq \emptyset \quad \text{for } i = 1, 2, \ldots, n \]

holds and since \( F(t) \) has exactly \( n \) values we have

\[ F(t) \subset B_\epsilon(F(a)). \]

By (1) \( F(a) \subset B_\epsilon(F(t)) \) holds. So if \( t \) is in \( U \) then

\[ H(F(t), F(a)) < \epsilon. \]

\( \square \)

**Lemma 7.14.** Let \( X \) be a topological space. Let \( Y \) be a Hausdorff topological space. Let \( F : X \rightharpoonup Y \) be a l.s.c. multifunction and let \( f : X \rightarrow Y \) be a continuous function. Let for each \( x \in X \) the set \( F(x) \setminus \{f(x)\} \) be nonempty. Then the multifunction \( G : X \rightharpoonup Y \) defined by \( G(x) = F(x) \setminus \{f(x)\} \) is l.s.c.
Proof. Let \( x \) be a fixed point of \( X \). Let \( U \) be an open set in \( Y \) and let \( G(x) \cap U \neq \emptyset \).

First we see that there exists \( t \) in \( F(x) \) such that \( t \neq f(x), \ t \in U \). Since \( t \neq f(x) \) there exist two disjoint open sets \( W, V \) in \( Y \) such that \( t \in W \subset U \) and \( f(x) \in V \). Since \( F(x) \cap W \neq \emptyset \) there exists an open neighbourhood \( O_1 \) of \( x \) such that for each \( s \) in \( O_1 \) \( F(s) \cap W \neq \emptyset \). From continuity of \( f \) it follows that there is an open neighbourhood \( O_2 \) of \( x \) such that for each \( z \) in \( O_2 \) \( f(z) \in V \). Denote \( O = O_1 \cap O_2 \). Then for each \( s \) in \( O \): \( F(s) \cap W \neq \emptyset \) and \( f(s) \in W \) therefore

\[
x \in O \subset G^+(W) \subset G^-(U).
\]

Hence \( G^-(U) \) is a neighbourhood of \( x \). \( \square \)

THEOREM 7.15. Let \( n > 0 \) be an integer. Let \( X \) be a topological space. Let \( (Y, d) \) be a metric space. Let \( F : X \rightharpoonup Y \) be a l.s.c. multifunction with exactly \( n \) values for each \( x \in X \). Let

\( (v) \) there exists a uniformly continuous function \( v : Y \to \mathbb{R} \) such that for each \( x \) in \( X \) the set \( v(F(x)) \) has exactly \( n \) elements (so \( v \) is one-to-one on \( F(x) \) holds).

Then there exist continuous functions \( f_i : X \to Y \) for \( i = 1, 2, \ldots, n \) such that for each \( x \in X \)

\[
F(x) = \{f_1(x), f_2(x), \ldots, f_n(x)\}
\]

holds.

Proof.

(1) If \( n = 1 \) then Theorem 7.15 is true.

(2) Let \( n > 0 \) be an integer and let us suppose that Theorem 7.15 is valid for each integer \( k: 0 < k < n \). By Lemma 7.13 the multifunction \( F \) is Hausdorff continuous and the conditions of Theorem 7.11 are fulfilled. Therefore there exists a continuous selection for \( F \). Let us denote this selection by \( f_1 \). Let us define a multifunction \( G : X \rightharpoonup Y \) as follows:

\[
G(x) = F(x) \setminus \{f_1(x)\}
\]

for each \( x \in X \). By Lemma 7.14 the multifunction \( G \) is l.s.c. and we can see that it has exactly \( n - 1 \) values. Since \( G(x) \subset F(x) \) for each \( x \in X \) the condition \( (v) \) of Theorem 7.15 is fulfilled for \( G \). Hence, since Theorem 7.15 is valid for a multifunction with \( n - 1 \) values, there exist \( n - 1 \) continuous functions \( f_2, f_3, \ldots, f_n \) from \( X \) into \( Y \) such that for each \( x \in X \)

\[
G(x) = \{f_2(x), f_3(x), \ldots, f_n(x)\}
\]

holds. Therefore for each \( x \in X \)

\[
F(x) = \{f_1(x)\} \cup G(x) = \{f_1(x), f_2(x), \ldots, f_n(x)\}.
\]

\( \square \)
7.3 Continuous selections for Lipschitz multifunctions

As we have already seen, in general, there is no guarantee that a "nice" multifunction will have a continuous selection. As we show later in this chapter, even closed-valued continuous multifunctions defined on a compact interval and with values in \( \mathbb{R} \) need not have a continuous selection. But in this section we show that if such a multifunction is locally Lipschitz, it does have a continuous selection.

**Definition 7.16.** If \( K \) is a positive real number, and \((X,d),(Y,\rho)\) are metric spaces, we say that a multifunction \( F \) from \( X \) to \( Y \) is \( K \)-Lipschitz if for every \( x_1, x_2 \) from \( X \) the inequality

\[
H_\rho(F(x_1), F(x_2)) \leq Kd(x_1, x_2)
\]

is true. (By \( H_\rho \) we denote a Hausdorff metric on \( 2^Y \setminus \{\emptyset\} \) derived in a natural way from \( \rho \).)

In what follows we will use the following technical lemmas:

**Lemma 7.17.** Let \( Y \) be a Banach space over \( \mathbb{R} \). Let \( a \in \mathbb{R} \), let \( m \) be a positive real number. Let \( I = [a,a+m] (I = [a-m,a]) \subset \mathbb{R} \). Let \( F : I \rightarrow Y \) be a \( K \)-Lipschitz multifunction. Let \( r > 0 \), \( r < K \). Let \( b \in F(a) \). Then there exists an \( M \)-Lipschitz function \( f : I \rightarrow Y \) such that \( M = (K + r) \), \( f(a) = b \) and for each \( x \) in \( I \)

\[
d(f(x), F(x)) = \inf\{d(f(x), t) \mid t \in F(x)\} < r.
\]

Moreover, \( f(I) \subseteq B(b, 2Km) \) holds.

**Proof.** Let us consider the case \( I = [a,a+m] \). The case \( I = [a-m,a] \) is symmetrical.

Let \( n \in \mathbb{N} \) be such that

\[
K \frac{m}{n} < \frac{r}{6} \quad \text{and} \quad \frac{m}{n} < \frac{1}{3}.
\]

Let us define

\[
x_i = a + \frac{m}{n} \quad \text{for} \quad i = 0, 1, 2, \ldots, n.
\]

Denote \( b = y_0 \). Since \( F \) is \( K \)-Lipschitz, there exists a point \( y_1 \in F(x_1) \) such that

\[
d(y_0, y_1) \leq H(F(x_0), F(x_1)) + \frac{rm}{2n} \leq Kd(x_0, x_1) + \frac{rm}{2n} \leq K \frac{m}{n} + \frac{rm}{2n} \leq \left( K + \frac{r}{2} \right) \frac{m}{n}.
\]

By final induction we can find a set \( \{y_0, y_1, \ldots, y_n\} \) such that \( \forall i = 0, 1, 2, \ldots, n, y_i \in F(x_i) \) and

\[
d(y_i, y_{i+1}) \leq \left( K + \frac{r}{2} \right) \frac{m}{n} \quad \text{for} \quad i \leq n - 1.
\]

Let us define a continuous function \( f : [a,a+m] \rightarrow Y \) in this way: \( f(x_i) = y_i, i = 0, 1, 2, \ldots, n \)

\[
f(x) = \frac{1}{m} \left[ n(x - x_i)y_{i+1} + n(x_{i+1} - x)y_i \right] \quad \text{if} \quad x \in (x_i, x_{i+1}).
\]

We will prove that \( f \) is \( \left( K + \frac{r}{2} \right) \)-Lipschitz on \([a,a+m]\).
Let $x, x' \in [x_i, x_{i+1}]$, for some $i \in \{0, 1, \ldots, n\}$, $x < x'$. We obtain
\[
d(f(x), f(x')) = \frac{1}{m} \left| n(x' - x_i)y_{i+1} + n(x_{i+1} - x')y_i - n(x-x_i)y_{i+1} - n(x_{i+1} - x)y_i \right| \\
= \frac{n}{m} \left| (x' - x)y_{i+1} - (x - x')y_i \right| \leq \frac{n}{m} |x' - x| \cdot ||y_{i+1} - y_i|| \\
\leq \frac{n}{m} |x' - x| \left( K + \frac{r}{2} \right) \frac{m}{n} \leq \left( K + \frac{r}{2} \right) |x' - x| .
\]

(ii) In general, if $x < x_i < x_{i+1} \ldots, x_{i+k} < x'$ for some $i, k \in \{0, 1, \ldots, n\}$, $i+k < n$ then, because of (i)
\[
d(f(x), f(x')) \\
\leq d(f(x), f(x_i)) + d(f(x_i), f(x_{i+1})) + \ldots + d(f(x_{i+k-1}), f(x_{i+k})) + d(f(x_{i+k}), f(x')) \\
\leq \left( K + \frac{r}{2} \right) |x_i - x| + \left( K + \frac{r}{2} \right) |x_{i+1} - x_i| + \ldots + \left( K + \frac{r}{2} \right) |x' - x_{i+k}| \\
= \left( K + \frac{r}{2} \right) |x' - x| .
\]
Now, let $x \in [a, a+m]$, then $x \in [x_i, x_{i+1}]$ for some $i \in \{0, 1, \ldots, n\}$. So
\[
d(f(x), F(x)) = \inf \{d(f(x), t) \mid t \in F(x)\} \\
= \inf \left\{ \left\| \frac{n}{m} (x - x_i)y_{i+1} + \frac{n}{m} (x_{i+1} - x)y_i - t \right\| \mid t \in F(x) \right\}
\]
Since $F$ is $K$-Lipschitz there exists a point $p$ from $F(x)$ such that $d(p, y_{i+1}) \leq \left( K + \frac{r}{2} \right) (x_{i+1} - x)$ therefore
\[
d(f(x), p) \leq d(f(x), y_i) + d(y_i, y_{i+1}) + d(y_{i+1}, p) \\
\leq \left( K + \frac{r}{2} \right) (x - x_i) + \left( K + \frac{r}{2} \right) \frac{m}{n} + \left( K + \frac{r}{2} \right) (x_{i+1} - x) \\
\leq \left( K + \frac{r}{2} \right) (x_{i+1} - x_i) + \left( K + \frac{r}{2} \right) \frac{m}{n} \leq 2 \left( K + \frac{r}{2} \right) \frac{m}{n} \leq 2 \frac{r}{6} + \frac{r}{n} < r
\]
so $d(f(x), F(x)) < r$ for each $x$ from $[a, a+m]$. Now, since $f(a) = b$ and $f$ is a $(K+r)$-Lipschitz function, for $r$ such that $r < K$ and for each $x$ from $[a, a+m]$ we have
\[
d(b, f(x)) = d(f(a), f(x)) \leq (K + r)|x - a| \leq 2K|a + m - a| \leq 2Km
\]
so $f([a, a+m]) \subseteq B(b, 2Km)$.

\textbf{Lemma 7.18.} Let $B$ be a finitely dimensional Banach space. Let $a \in \mathbb{R}$, let $l$ be a positive real number. Let $I = [a, a+l] \setminus ([a-l, a])$. Let $F : I \rightarrow B$ be a $K$-Lipschitz multifunction with closed values. Then $F$ has a $K$-Lipschitz selection on $I$. 

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Proof. We will prove the Theorem only for the case \( I = [a, a + l] \). According to Lemma 7.12 there exists a sequence \( \{f_i\}_{i=1}^{\infty} \) of functions \( f_i : [a, a + l] \to B \) such that for each index \( i \) from \( \mathbb{N} \) and each \( x \) from \( [a, a + l] \) \( d(f_i(x), F(x)) < \frac{1}{i} \) is true. Moreover, each function \( f_i \) is \( (K + \frac{1}{i}) \)-Lipschitz and

\[
f_i([a, a + l]) \subset B(b, 2KI).
\]

This implies that for every \( x \) from \( X \) the set \( \{f_i(x) \mid i = 1, 2, \ldots \} \) is precompact. Since \( B \) is finitely dimensional, according to Arzela-Ascoli theorem the set

\[
M = \{f_i \mid i \in 1, 2, \ldots \}
\]

is precompact. So there exists a continuous function \( f : [a, a + l] \to B \) such that \( f \) is a uniform limit of a sequence \( \{f_i\}_{i=1}^{\infty} \) (a subsequence of \( \{f_i\}_{i=1}^{\infty} \)) of functions from \( M \).

Let us consider an \( \epsilon > 0 \). As we have proved above there exists an index \( k \) such that \( f_j \) is \( (K + \epsilon) \)-Lipschitz for each \( j \geq k \). That means that the function \( f \) is also \( (K + \epsilon) \)-Lipschitz. \( f \) is proved to be \( K \)-Lipschitz.

Now it is simple to realize that \( f \) is a selection of \( F \). For each \( \epsilon > 0 \) there exists an index \( m \) such that for each \( x \) from \( X \)

\[
d(f_m(x), F(x)) < \epsilon \quad \text{and} \quad \sup_{x \in [a, a+l]} |f_m(x) - f(x)| < \epsilon.
\]

So for every \( x \) from \( X \) \( d(f(x), F(x)) < 2\epsilon \). Since \( \epsilon \) was an arbitrary positive real number, for each \( x \) from \( X \) \( d(f(x), F(x)) = 0 \) is true. \( F \) has closed values so \( f \) is a selection of \( F \). \( \square \)

Now we are prepared to prove our first theorem concerning Lipschitz multifunctions.

**Theorem 7.19.** Let \( B \) be a finitely dimensional Banach space over \( \mathbb{R} \). Let \( F : \mathbb{R} \rightharpoonup B \) be a \( K \)-Lipschitz multifunction with closed values. Then \( F \) has a \( K \)-Lipschitz selection on \( \mathbb{R} \).

Proof. This is a simple consequence of Theorem 7.11 so we will only give an outline of the proof. Let \( b \) be an element of the set \( F(0) \). Using Theorem 7.11, we can define by induction \( K \)-Lipschitz selections \( f_1, f_2, \ldots, f_{2i}, f_{2i+1}, \ldots \) of \( F \) such that for each nonnegative integer \( i \) the function \( f_{2i} \) (\( f_{2i+1} \)) is defined on \([2i, 2i+2)\) \((-2i-2, -2i)\) and \( f_{2i}(2i+2) = f_{2i+1}(2i+2) \) \((f_{2i+1}(-2i-2) = f_{2i+1}(-2i-2))\) and such that \( f_i(0) = f_2(0) = b \). It is easy to see that a function \( f : \mathbb{R} \to B \) defined by \( f(x) = f_{2i}(x) \) if \( x \in [2i, 2i+2] \) and \( f(x) = f_{2i+1}(x) \) if \( x \in [-2i-2, -2i] \) is correctly defined and it is a \( K \)-Lipschitz selection of \( F \). \( \square \)

The above theorem is true for certain multifunctions with non-convex and non-compact values. It is a generalization of the following result obtained by Guričan and Kostyrko in [16] in 1985:

**Corollary 7.20.** Let \( n \) be a positive integer, let \( B = \mathbb{R}^n \). Let \( F : \mathbb{R} \rightharpoonup B \) be a \( K \)-Lipschitz multifunction with convex compact (and non-void) values. Then \( F \) has a \( K \)-Lipschitz selection on \( \mathbb{R} \).
In the following lemma we shall use the following assumption concerning a multifunction \( F \) from \( \mathbb{R} \) to a Banach space \( B \):

**Assumption LFD.** For every \( x \) from \( \mathbb{R} \) there exists an open neighbourhood \( O(x) \subset \mathbb{R} \) and a finitely dimensional set \( B_x \subset B \) such that \( F(O(x)) \subset B_x \).

We say that a multifunction \( F : \mathbb{R} \rightarrow B \) is **locally Lipschitz** if for every real \( x \) there exists an open interval \( U_x \) and a positive real constant \( K_x \) such that \( x \in U_x \) and \( F \) is \( K_x \)-Lipschitz on \( U_x \).

**Lemma 7.21.** Let \( B \) be a Banach space. Let \( F : \mathbb{R} \rightarrow B \) be a locally Lipschitz multifunction with closed values. Let \( F \) satisfy the assumption LFD. Let \( a, b \in \mathbb{R} \) and \( b \in F(a) \). Then for every real \( c, d, c < d \) satisfying \( c \leq a \leq d \) there exists a Lipschitz selection \( f : [c, d] \rightarrow B \) of \( F \) such that \( f(a) = b \).

**Proof.** It suffices to show that \( F \) is Lipschitz on \([c, d]\) and that there exists a finitely dimensional subset \( Z \) of \( B \) such that \( F([c, d]) \subset Z \). After that we can apply Theorem 7.11.

We proceed by a usual "locally on compact implies globally on compact" procedure. Obviously for every \( x \) from \([c, d]\) there exists an open interval \( U_x \), a positive real number \( K_x \) and a finitely dimensional subset \( B_x \) of \( B \) such that \( x \in U_x \), \( F(U_x) \subset B_x \) and \( F \) is \( K_x \)-Lipschitz on \( U_x \).

Consider the following open cover \( C \) of \([c, d]\):

\[
C = \{ U_x \mid x \in [c, d] \}.
\]

There exists a finite subcover \( S \) of \( C \) and a positive integer \( n \) such that

\[
S = \{ U_{x_1}, U_{x_2}, \ldots, U_{x_n} \}.
\]

Let us denote

\[
M = \max\{K_{x_1}, K_{x_2}, \ldots, K_{x_n}\}.
\]

Then \( F \) is \( M \)-Lipschitz on each interval \( U_{x_i} \) for \( i \in \{1, 2, \ldots, n\} \). The fact \([c, d] \subset U := \bigcup_{i=1}^{n} U_{x_i}\) implies \( F \) is \( M \)-Lipschitz on \([c, d]\).

Moreover,

\[
F([c, d]) \subset F(U) \subset Z := \bigcup_{i=1}^{n} B_{x_i},
\]

and we can see that \( Z \) is finitely dimensional.

If \( c < a < d \) Theorem 7.11 implies \( F \) has an \( M \)-Lipschitz selection \( h ( g ) \) on \([c, a] \) ([\( a, d]\)) such that \( g(a) = h(a) = b \). So if \( c < a < d \) the function \( f : [c, d] \rightarrow B \) defined by \( f(x) = g(x) \) on \([c, a]\) and \( f(x) = h(x) \) on \([a, d]\) is a Lipschitz selection of \( F \) on \([c, d]\). The proof for the case \( a = c, a = d \) is even easier. \( \square \)

Now we are prepared to formulate and prove the main theorem of this section.
Theorem 7.22. Let $B$ be a Banach space over $\mathbb{R}$. Let $F : \mathbb{R} \rightrightarrows B$ be a locally Lipschitz multifunction with closed values. Let $F$ satisfy the assumption LFD. Let $a \in \mathbb{R}$ and $b \in F(a)$. Then $F$ has a continuous selection $f$ on $\mathbb{R}$ such that $f(a) = b$.

To realize that the assumptions of our final result, Theorem 7.22, can hardly be weakened, compare the following three assertions:

1. There exists a finitely valued Lipschitz multifunction from a unit circle into $\mathbb{R}^2$ that has no continuous selection. (In fact Example 7.10 can illustrate this situation. Of course, each multifunction with values in $\mathbb{R}^2$ or $\mathbb{R}$ automatically satisfies the assumption LFD.)

2. There exists a Hausdorff continuous multifunction from the compact interval $[-1, 0]$ to $\mathbb{R}$ with closed values, which is locally Lipschitz in every point of $[-1, 0)$ and has no continuous selection. (See Example 7.23.)

3. Each locally Lipschitz multifunction with closed values from $\mathbb{R}$ to a Banach space, satisfying the assumption LFD has a continuous selection. This is what our Theorem 7.22 claims.

Before presenting the proof of Theorem 7.22 let us examine the following example, which is a construction of a Hausdorff continuous multifunction $S : [-1, 0] \rightrightarrows \mathbb{R}$ with closed values, which is locally Lipschitz on $[-1, 0)$ and has no continuous selection. We will use this example also in the section 7.4.

Example 7.23. Let $S : [-1, 0] \rightrightarrows \mathbb{R}$ be defined as follows:

$$S(0) = \mathbb{R},$$

$$S(x) = \left\{ \frac{n(n+1)}{2} x + \frac{k}{2^n} \mid k \in \mathbb{Z} \right\} \cup \left\{ \frac{n(n+1)}{2^{n+1}} x + \frac{n+1}{2^{n+1}} + \frac{k}{2^n} \mid k \in \mathbb{Z} \right\}$$

for every positive integer $n$ and every $x \in \left[ -\frac{1}{n}, -\frac{1}{n+1} \right]$. In other words: the intersection of the graph of $S$ with the set $\left[ -\frac{1}{n}, -\frac{1}{n+1} \right] \times \mathbb{R}$ is the system of segments joining the following couples of points: the point $\left[ -\frac{1}{n}, \frac{m}{2^n} \right]$ with the point $\left[ -\frac{1}{n+1}, \frac{m}{2^n} + \frac{1}{2} \right]$ and $\left[ -\frac{1}{n}, \frac{m}{2^n} \right]$ with the point $\left[ -\frac{1}{n+1}, \frac{m}{2^n} + \frac{1}{2} + \frac{1}{2^{n+1}} \right]$ where $m$ is an arbitrary integer.

Of course, $S$ is Hausdorff continuous on $[-1, 0)$; so, it is l.s.c. on this set. Now, it suffices to show that $S$ is Hausdorff continuous in 0. But it is easy to see that for every $t \in \left[ -\frac{1}{n}, 0 \right)$ the following holds: if $s \in S(t)$ then $s + \frac{k}{2^n} \in S(t)$ for every integer $k$, so

$$H(S(t), \mathbb{R}) \leq \frac{1}{2^n},$$

where $H$ denotes the Hausdorff metric defined on $2^\mathbb{R}$.

To show that $S$ is locally Lipschitz on $[-1, 0)$ it is sufficient to show that it is $n(n+1)$-Lipschitz on
$I_n = \left[ -\frac{1}{n}, -\frac{1}{n+1} \right]$ for every $n \in \mathbb{N}$, $n > 0$.

Let $x_1, x_2 \in I_n$. Let $y_1 \in S(x_1)$. Then there exists an integer $k$ such that

$$y_1 = \frac{n(n+1)}{2} x_1 + \frac{k}{2^n} \quad \text{or} \quad y_1 = n(n+1) \frac{2^n+1}{2^{n+1}} x_1 + \frac{n+1}{2^{n+1}} + \frac{k}{2^n}.$$ 

There exists also $y_2$ from $S(x_2)$ such that

$$y_2 = \frac{n(n+1)}{2} x_2 + \frac{k}{2^n} \quad \text{or} \quad y_2 = n(n+1) \frac{2^n+1}{2^{n+1}} x_2 + \frac{n+1}{2^{n+1}} + \frac{k}{2^n}$$

so $|y_1 - y_2|$ equals

$$\frac{n(n+1)}{2} |x_1 - x_2| \quad \text{or} \quad \frac{n(n+1)(2^n+1)}{2^{n+1}} |x_1 - x_2|.$$

In both cases we have

$$|y_1 - y_2| \leq K_n |x_1 - x_2|, \quad \text{where} \quad K_n = n(n+1). \tag{7.7}$$

In the same way we can pick a point $y_2$ from $S(x_2)$ first and find a point $y_1$ from $S(x_1)$ such that the inequality (7.7) is true.

This means that for each $x_1, x_2$ from $I_n$

$$H(S(x_1), S(x_2)) \leq K_n |x_1 - x_2|$$

is true. We have proved that $S$ is locally Lipschitz on $[-1, 0]$.

It is easy to see that $S$ has no continuous selection on $[-1, 0]$: every continuous selection $f$ of $S$ defined on the interval $[-1, 0]$ has the property

$$\lim_{t \to 0^+} f(t) = +\infty.$$

Now we will prove the main theorem of this section.

**Proof of Theorem 7.22.** For $n = 1, 2, 3 \ldots$ denote $I_n \in [-n, n]$. In what follows we will proceed by induction. Let us suppose, without loss of generality, that $a = 0$.

1. According to Lemma 7.13 there exists a Lipschitz selection $f_1 : T_1 \to B$ of $F$ on the interval $I_1$ such that $f(a) = b$. Let us denote $f_1(-1) = b_1$ and $f_1(1) = c_1$.

2. Suppose that for $n \in \mathbb{N}$, $n = 1, 2, \ldots k$ there exist Lipschitz selections $f_n$ of $F$ on $I_n$ such that if $l, m \in \{1, 2, \ldots k\}$, $l > m$ then $f_l(x) = f_m(x)$ for each $x$ from $I_m$.

For each of the $n$ considered let us denote $f_n(-n) = b_n$ and $f_n(n) = c_n$.

Since $b_k \in F(-k)$ there exists a Lipschitz selection $g_k$ of $F$ on $[-k - 1, -k]$ such that $g_k(-k) = b_k$. 

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Since $c_k \in F(k)$ there exists a Lipschitz selection $h_k$ of $F$ on $[k, k+1]$ such that $h_k(k) = c_k$.

Let us define a function $f_k$ on $I_k$ by

\[
\begin{align*}
    f_k(x) &= g_k(x) \quad \text{for } x \text{ from } [-k - 1, -k], \\
    f_k(x) &= f_{k-1}(x) \quad \text{for } x \text{ from } [-k, k], \\
    f_k(x) &= h_k(x) \quad \text{for } x \text{ from } [k, k + 1].
\end{align*}
\]

We have just constructed by induction a sequence of Lipschitz selections $f_k$ of $F$ on the intervals $I_k$ such that if $k_1 < k_2$ then $f_{k_2}(x) = f_{k_1}(x)$ for all $x$ from $I_{k_1}$. All functions $f_k$ are continuous selections of $F$ on their domains.

Let us define a function $f : \mathbb{R} \to B$ by

\[
\begin{align*}
    f(x) &= f_1(x) \quad \text{for } x \in [-1, 1], \\
    f(x) &= f_k(x) \quad \text{for } x \in [-k - 1, -k] \cup [k, k + 1], k = 1, 2, \ldots
\end{align*}
\]

The function $f$ is a selection of $F$ on $\mathbb{R}$. It is continuous because all functions $f_k$ are continuous. □

## 7.4 Example

This section presents a very important example. It was published for the first time in 1996 in [23] and it shows that even a very nice multifunction need not have a continuous selection.

**Example 7.24.** Let us consider the multifunction $S$ from Example 7.23. The multifunction $S$ is not u.s.c. To see this, define a set

\[
U = \bigcup_{k \in \mathbb{Z}} \left( \frac{k}{2} - \frac{1}{2^{k+1}}, \frac{k}{2} + \frac{1}{2^{|k|}} \right).
\]

Then $U$ is an open neighbourhood of the set $S(-1)$ and for every neighbourhood $V$ of the point $-1$ there exists $t \in V$ such that $S(t)$ is not a subset of $U$. A problem of this kind will not appear when we make the set $\mathbb{R} - S(x)$ "sufficiently small", i.e., a subset of a compact interval.

Let $G : [-1, 0) \to \mathbb{R}$ be defined as follows:

\[
G(x) = \left( -\infty, \frac{1}{x} \right] \cup \left[ -\frac{1}{x}, +\infty \right) \quad \text{for } x \in (-\infty, 0).
\]

Now, let us define $F : [-1, 0) \to \mathbb{R}$ as follows:

\[
\begin{align*}
    F(x) &= S(x) \cup G(x) \quad \text{for } x \in [-1, 0) \\
    F(0) &= S(0) = \mathbb{R}.
\end{align*}
\]

It is easy to verify that $F$ is u.s.c. and Hausdorff continuous at the point 0.

Since both $S$ and $G$ are Hausdorff continuous on the set $[-1, 0)$, $F = S \cup G$ is Hausdorff continuous,
too. $F$ is u.s.c. on $[-1, 0)$. For example let $x \in \left[ -\frac{1}{n}, -\frac{1}{n+1} \right]$ and let $W$ be an open neighbourhood of the set $F(x)$.

Let us denote

$$A = F(x) - \left( (-\infty, -\frac{1}{x}) \cup \left( -\frac{1}{x}, +\infty \right) \right).$$

Let

$$A(\alpha) = \bigcup_{a \in A} (a - \alpha, a + \alpha) \quad \text{for} \quad \alpha > 0.$$

Then there exists an $\epsilon > 0$ such that the set

$$Z = (-\infty, \frac{1}{x} + \epsilon) \cup (-\frac{1}{x} - \epsilon, +\infty) \cup A(\epsilon)$$

is a subset of $W$. Let $I$ be the set of such indices $k \in \mathbb{N}$, that there exists $t \in \left[ -\frac{1}{n}, -\frac{1}{n+1} \right]$ for which the set

$$\left\{ \frac{n(n+1)}{2} t + \frac{k}{2^n}, n(n+1) \frac{2^n + 1}{2^{n+1}} t + \frac{k}{2^{n+1}} \right\} \cap [-n - 1, n + 1]$$

is nonempty.

Each of the functions

$$\frac{1}{x}, \frac{1}{x}, \frac{n(n+1)}{2} x + \frac{k}{2^n}, \text{ and } n(n+1) \frac{2^n + 1}{2^{n+1}} x + \frac{n + 1}{2^{n+1}} + \frac{k}{2^n} \quad (k \in I)$$

is uniformly continuous on the interval $[-\frac{1}{n}, -\frac{1}{n+1}]$. The set $I$ is finite. So, considering the form of the set $F(x)$, it is easy to see that there exists an $\delta > 0$ (i.e. $\delta = \frac{\epsilon}{2(n+1)^2}$) such that for every $t \in \mathbb{R}$ satisfying $|t - x| < \delta$, $F(t) \subset Z \subset W$ holds.

So, $F$ is Hausdorff continuous, l.s.c. and u.s.c. on the interval $[-1, 0]$. Of course, $F$ has no continuous selection on $[-1, 0]$.

### 7.5 Selection Problems and Extension Problems

We say that a mapping $f : X \to Y$ from a set $X$ into a set $Y$ is an extension of a mapping $g : A \to Y$ from a subset $A \subset X$ into $Y$ if the restriction $f|_A$ coincides with $g$. If $X$ and $Y$ are topological spaces, $A \subset X$ is a closed subset of $X$ and $g$ is a continuous function defined on $A$, we can define the following multifunction $F : X \rightharpoonup Y$:

$$F(x) = Y \quad \text{if} \quad x \not\in A,$$

$$F(x) = \{g(x)\} \quad \text{if} \quad x \in A.$$

We see immediately that $g$ has a continuous extension on $X$ if and only if $F$ has a continuous selection $f : X \to Y$. In general, every extension problem is a partial case of a selection problem. So every selection theorem implies results concerning an extension problem.
Exercises

Exercise 1 Find a l.s.c. multifunction $F : \mathbb{R} \rightharpoonup \mathbb{R}$ that has no continuous selections.
Exercise 2 Find a u.s.c. multifunction $F : \mathbb{R} \rightharpoonup \mathbb{R}$ that has no continuous selections.
Exercise 3 Prove that a continuous multifunction $F : \mathbb{R} \rightharpoonup \mathbb{R}$ with compact values has a continuous selection.
Exercise 4 Do Exercises 1 and 2 again, this time $F$ has to be a finite-valued multifunction.
Exercise 5 Define a continuous multifunction with exactly three values for each $x$, such that it has no continuous selection. (Hint: study Example 7.10)