Chapter 5

Hyperspaces

In this chapter we will introduce some topologies on the closed subsets of a topological (or metric) space. We follow closely the approach of Beer, based on an interplay between topologies on hyperspaces and weak topologies on spaces of functions. Almost all results presented here are taken from his work (see References).

If \((Y, \tau)\) is a metric space, we denote

\[
\text{cl}(Y) = \{ A \subset Y : Y \text{ is closed, } Y \neq \emptyset \}
\]

\[
2^Y = \text{cl}(Y) \cup \{\emptyset\}.
\]

It is quite clear that a multifunction can be perceived as a function. If \((X, T), (Y, \tau)\) are topological spaces and \(F : X \rightarrow Y\) is a multifunction with closed, nonempty values, then we can define a function \(f : X \rightarrow \text{cl}(Y)\) by

\[
\forall x \in X \quad f(x) = F(x).
\]

(Now, every set \(F(x)\) is perceived as a point, as an element of \(\text{cl}(Y)\)).

Since the function theory is very rich in results, it is quite reasonable to study the structure of \(\text{cl}(Y)\) \((2^Y)\) equipped by a topology and to examine the properties of the function \(f\).

First, let us concentrate on this question. If \((Y, \tau)\) is a topological space, how would a "natural" topology on \(\text{cl}(Y)\) look like? Minimally, a hyperspace topology should extend the initial topology on the underlying topological space. In other words, if we restrict a hyperspace topology (a topology on \(\text{cl}(Y)\)) to the singleton subsets, we want the induced topology to agree with the initial topology on the underlying space. We call this property admissibility.

5.1 The notion of excess

Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\). For each \(B \subset X\) and for an \(\epsilon > 0\) define

\[
S_\epsilon(B) = \{ x \in X : d(x, B) < \epsilon \}.
\]

The excess of \(A\) over \(B\) with respect to \(d\) is defined by the formula

\[
e_d(A, B) = \sup\{d(a, B) : a \in A\}.
\]

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The excess of $A$ over $B$ may well be $+\infty$; for example, this will occur if $B$ is bounded and $A$ is unbounded. Keeping in mind that the infimum of an empty set of reals is $+\infty$, it is clear that

$$e_d(A, B) = \inf\{\epsilon > 0 : S_\epsilon[B] \supset A\}.$$ 

Excess so defined is not symmetric. For example, in the line with the usual metric, with $A = [0, 5]$ and $B = [5, 6]$, we have $e_d(A, B) = 5$ and $e_d(B, A) = 1$. We adopt the convention that if $A$ is nonempty, then $e_d(\emptyset, A) = 0$.

The gap $D_d(A, B)$ between nonempty subsets $A$ and $B$ of a metric space $(X, d)$ is given by

$$D_d(A, B) = \inf\{d(a, B) : a \in A\} = \inf\{d(b, A) : b \in B\}$$

$$= \inf\{d(a, b) : a \in A, b \in B\}$$

$$= \inf\{\epsilon > 0 : A \bigcap S_\epsilon[B] \neq \emptyset\}.$$ 

We can see that $D_d([0, 5], [5, 6]) = 0$.

Unlike the excess functional, the gap functional is finite-valued and symmetric. Notice, that excess and gap reduce to ordinary distance when $A$ is a singleton subset.

The excess functional has the following interesting property:

**Lemma 5.1.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. Then $e_d(A, B) = \sup_{x \in X} d(x, B) - d(x, A)$.

*Proof.* We work with the formula $e_d(A, B) = \inf\{\epsilon > 0 : A \subset S_\epsilon[B]\}$. Let us write $\lambda = \sup_{x \in X} d(x, B) - d(x, A)$. First, we show $e_d(A, B) \leq \lambda$. If $\lambda = +\infty$, there is nothing further to do. Otherwise, fix $a \in A$ and $\alpha > \lambda$. We have

$$d(a, B) = d(a, B) - d(a, A) \leq \lambda < \alpha,$$

and it follows that $a \in S_\alpha[B]$. Thus $A \subset S_\alpha[B]$. This shows that $\inf\{\epsilon > 0 : A \subset S_\epsilon[B]\} \leq \lambda$, and we obtain $e_d(A, B) \leq \lambda$.

We now show that $e_d(A, B) \geq \lambda$. Let $\epsilon > 0$ be arbitrary. Fix $x \in X$, and choose $a \in A$ with $d(x, a) < d(x, A) + \epsilon/2$. Now pick $b \in B$ with $d(a, b) < d(a, B) + \epsilon/2 \leq e_d(A, B) + \epsilon/2$. We have

$$d(x, B) \leq d(x, b) \leq d(x, a) + d(a, b) < d(x, A) + e_d(A, B) + \epsilon,$$

so that

$$d(x, B) - d(x, A) \leq e_d(A, B) + \epsilon.$$

Since $x$ was arbitrary, we obtain

$$\sup_{x \in X} d(x, B) - d(x, A) \leq e_d(A, B) + \epsilon.$$ 

Since $\epsilon$ was arbitrary, we have $\lambda \leq e_d(A, B)$, completing the proof. □
5.2 The Hausdorff metric

For metric spaces \((X, d)\) and \((Y, d')\) let us denote the topology of uniform convergence on \(C(X, Y)\) by \(\tau_{uc}\). There is a natural infinite-valued compatible metric for this function space given by

\[
d_{uc}(f, g) = \sup_{x \in X} d'(f(x), g(x)).
\]

It is a basic fact of elementary analysis that this uniform metric is complete whenever \((Y, d')\) is a complete metric space.

**Definition 5.2.** Let \((X, d)\) be a metric space. The Hausdorff metric topology \(\tau_{Hd}\) on \(cl(X)\) is the topology that \(cl(X)\) inherits from \((C(X, \mathbb{R}), \tau_{uc})\) under the identification \(A \leftrightarrow d(\cdot, A)\). For any two elements \(A_1\) and \(A_2\) of \(cl(X)\) we define

\[
H_d(A_1, A_2) = \sup_{x \in X} |d(x, A_1) - d(x, A_2)|.
\]

We say that \(H_d(A_1, A_2)\) is the Hausdorff distance between \(A_1\) and \(A_2\).

The (in general infinite-valued) metric \(H_d\) is called the Hausdorff metric on \(cl(X)\).

**Remark 5.3.** The Hausdorff metric convergence can be expressed in terms of excess. By Lemma 5.1 we have for \(A_1, A_2 \in cl(X)\)

\[
e_d(A_1, A_2) = \sup_{x \in X} d(x, A_2) - d(x, A_1)
\]

\[
e_d(A_2, A_1) = \sup_{x \in X} d(x, A_1) - d(x, A_2)
\]

and this implies

\[
H_d(A_1, A_2) = \sup_{x \in X} |d(x, A_1) - d(x, A_2)| = \\
= \max\{e_d(A_1, A_2), e_d(A_2, A_1)\} = \\
= \inf\{\epsilon > 0 : S_\epsilon(A_1) \supseteq A_2 \text{ and } S_\epsilon(A_2) \supseteq A_1\} = \\
= \max\{\sup_{x \in A_2} d(x, A_1), \sup_{x \in A_1} d(x, A_2)\}.
\]

Observe that the map \(x \mapsto \{x\}\) is an isometry of \((X, d)\) into \((cl(X), H_d)\) and that Hausdorff distance is a finite-valued metric on the closed and bounded subsets \(clb(X)\) of \(X\). A countable base for a uniformity on \(cl(X)\) compatible with the Hausdorff metric topology consists of all sets of the form

\[
\{(A, B) : A \subset S_{\frac{1}{n}}(B) \text{ and } B \subset S_{\frac{1}{n}}(A)\} \quad \text{for} \quad n = 1, 2, 3, \ldots
\]

Speaking about the Hausdorff metric topology, we would like to mention briefly the so-called Attouch-Wets topology. If \((X, d)\) is a metric space, the Attouch-Wets topology \(\tau_{AWd}\) on \(cl(X)\) is the topology that \(cl(X)\) inherits from \(C(X, \mathbb{R})\), equipped with the topology \(\tau_{ucb}\) of uniform convergence.
on bounded subsets of $X$, under the identification $A \leftrightarrow d(\cdot, A)$. It is not hard to show that the Hausdorff metric topology $\tau_{H_d}$ equals the Attouch-Wets topology $\tau_{AW_d}$ on $\text{cl}(X)$ if and only if $(X, d)$ is a bounded metric space.

Now we are going to concentrate on some properties of $(\text{cl}(X), H_d)$. It should not be surprising that they depend on the properties of the space $(X, d)$. First we need to introduce two technical assertions.

**Lemma 5.4.** Let $(X, d)$ be a metric space and let $f \in C(X, \mathbb{R})$ be in the closure of \{d(\cdot, f) : f \in \text{cl}(X)\} with respect to the topology of pointwise convergence. Suppose $A = \{x \in X : f(x) = 0\}$ is nonempty, and for each $x \in X$, $d(x, A) \leq f(x)$ is true. Then $f$ is the distance functional for the set $A$.

**Proof.** We must show that $f(x) \leq d(x, A)$ for each $x \in X$, from which it will follow that $f(x) = d(x, A)$ from our assumption $d(x, A) \leq f(x)$. Fix $x \in X$. Suppose to the contrary that $d(x, A) < f(x)$ holds. Set $\epsilon = f(x) - d(x, A)$ and pick $a \in A$ with

$$d(x, a) < d(x, A) + \epsilon / 3.$$ 

Since $f$ is a pointwise limit of distance functionals, there exists $B \in \text{cl}(X)$ such that $|f(a) - d(a, B)| < \epsilon / 3$. Since $f(a) = 0$, we obtain this inequality string:

$$f(x) < d(x, B) + \frac{\epsilon}{3} \leq d(x, a) + d(a, B) + \frac{\epsilon}{3} < d(x, a) + f(a) + \frac{2\epsilon}{3} < d(x, A) + \epsilon.$$

This contradicts the definition of $\epsilon$, completing the proof. \hfill \square

**Lemma 5.5.** Let $(X, d)$ be a complete metric space and let $\{A_n\}$ be a sequence in $\text{cl}(X)$. Suppose $\{d(\cdot, A_n)\}$ converges uniformly on bounded subsets of $X$ to $f \in C(X, \mathbb{R})$. Then $f$ is a distance functional for a nonempty closed subset $A$ of $X$.

**Proof.** Fix $x_1$ in $X$. Suppose $f(x_1) = \alpha$ and $\epsilon > 0$. According to Lemma 5.4 we establish the assertion of our lemma if we can produce $w \in X$ such that $f(w) = 0$ and $d(w, x_1) < \alpha + 2\epsilon$.

Since $\{d(\cdot, A_n)\}$ is uniformly Cauchy on $S_{\alpha+2\epsilon}[x_1]$, there exists a strictly increasing sequence $\{N_k\}$ of positive integers such that for all $m > n \geq N_k$, we have

$$\sup\{|d(x, A_m) - d(x, A_n)| : x \in S_{\alpha+2\epsilon}[x_1]\} < 2^{-k}\epsilon.$$

Since $d(x_1, A_{N_k}) \leq f(x_1) + 2^{-1}\epsilon = \alpha + 2^{-1}\epsilon$, we have

$$d(x_1, A_{N_k}) < \alpha + 2^{-1}\epsilon + 2^{-1}\epsilon = \alpha + \epsilon.$$

Thus, there exists $x_2 \in A_{N_k}$ with $d(x_1, x_2) < \alpha + \epsilon$. Note that $x_2 \in S_{\alpha+2\epsilon}[x_1]$; so,

$$d(x_2, A_{N_k}) = |d(x_2, A_{N_k}) - d(x_2, A_{N_k})| < 2^{-2}\epsilon.$$

Continuing, there exists $x_3 \in A_{N_k}$ with $d(x_2, x_3) < \epsilon / 4$ and we have $d(x_3, x_1) < \alpha + \epsilon + \epsilon / 4$. In this manner, we produce a sequence $\{x_n\}$ such that for each $n = 3, 4, \ldots$
5.2 The Hausdorff metric

(i) \( x_n \in A_N \);

(ii) \( d(x_n, x_1) < \alpha + \epsilon \left( 1 + \sum_{i=2}^{n-1} 2^{-i} \right) < \alpha + 2\epsilon - \frac{1}{2}\epsilon \);

(iii) \( d(x_{n+1}, x_n) < 2^{-n}\epsilon \).

By (iii) and completeness of \((X, d)\), the sequence \( \{x_n\} \) converges to some point \( w \in X \). By (ii), \( d(w, x_1) < \alpha + 2\epsilon \). Finally, by (i) and uniform convergence of \( \{d(\cdot, A_n)\} \) on \( \{w\} \cup \{x_n : n \in \mathbb{Z}^+\} \) to \( f \), we have

\[
 f(w) = \lim_{n \to \infty} d(x_n, A_{N_n}) = 0.
\]

By Lemma 5.4 the function \( f \) is a distance functional for some nonempty closed subset of \( X \). □

The following theorem speaks about the complete metrisability and compactness of the space \((\text{cl}(X), H_d)\).

**THEOREM 5.6.** Let \((X, d)\) be a metric space.

(1) \((\text{cl}(X), H_d)\) is complete if and only if \((X, d)\) is complete.

(2) \((\text{cl}(X), H_d)\) is totally bounded if and only if \((X, d)\) is totally bounded.

(3) \((\text{cl}(X), H_d)\) is compact if and only if \((X, d)\) is compact.

**Proof.** For (1) necessity follows from the fact that \( x \to \{x\} \) is an isometry and \( \{\{x\} : x \in X\} \) is closed in \((\text{cl}(X), H_d)\). Sufficiency follows from Lemma 5.5: \( \text{cl}(X) \) is a closed subset of the complete metric space \((C(X, \mathbb{R}), d_{\text{UC}})\) under the usual identification, and thus \((\text{cl}(X), H_d)\) is a complete metric space.

Necessity in (2) again follows from the isometric character of \( x \to \{x\} \). For sufficiency in (2), fix \( \epsilon > 0 \) and by total boundedness of \( X \) let \( F \) be a finite subset of \( X \) with \( X \subset S_\epsilon[F] \). For each closed subset \( A \) of \( X \) there exists a minimal subset \( E \) of \( F \) such that \( A \subset S_\epsilon[E] \), and it follows that the reverse inclusion \( E \subset S_\epsilon[A] \) must be satisfied. As a result, each closed subset of \( X \) has Hausdorff distance at most \( \epsilon \) from some subset of \( F \). Since \( F \) has only finitely many subsets and \( \epsilon > 0 \) was arbitrary, \((\text{cl}(X), H_d)\) is totally bounded.

Finally, a metric space is compact if and only if it is complete and totally bounded, from which assertion (3) follows. □

A characteristic feature of the Hausdorff metric topology is the continuity of excess and gap functionals with arbitrary fixed closed arguments. In the next result, we adopt the convention that "+∞ – +∞" is zero.
5.3 Weak topologies

**Theorem 5.7.** Let \((X, d)\) be a metric space and let \(B \subseteq \text{cl}(X)\). Then the functionals

\[
\begin{align*}
    e_d(B, \cdot) : (\text{cl}(X), H_d) &\to [0, +\infty], \\
    e_d(\cdot, B) : (\text{cl}(X), H_d) &\to [0, +\infty], \\
    D_d(B, \cdot) : (\text{cl}(X), H_d) &\to [0, +\infty)
\end{align*}
\]

are Lipschitz continuous with constant one.

**Proof.** We verify Lipschitz continuity only for the functional \(e_d(\cdot, B)\). Suppose \(A_0 \in \text{cl}(X)\), \(A_1 \in \text{cl}(X)\), and \(H_d(A_0, A_1)\) is finite. By the triangle inequality for excess (See Exercise 1), we have

\[
\begin{align*}
    e_d(A_0, B) &\leq e_d(A_0, A_1) + e_d(A_1, B), \\
    e_d(A_1, B) &\leq e_d(A_1, A_0) + e_d(A_0, B).
\end{align*}
\]

Since \(e_d(A_1, A_0) < +\infty\), it is clear that \(e_d(A_0, B) = +\infty\) if and only if \(e_d(A_1, B) = +\infty\). If both are finite then by subtraction of real numbers we have

\[
|e_d(A_1, B) - e_d(A_0, B)| \leq \max\{e_d(A_0, A_1), e_d(A_1, A_0)\} = H_d(A_0, A_1).
\]

\(\square\)

### 5.3 Weak topologies

**Definition 5.8.** Let \(\{(X_i, \tau_i) : i \in I\}\) be a family of Hausdorff spaces and let \(X\) be a nonempty set. Suppose that \(\mathcal{R} = \{f_i : i \in I\}\) is a family of functions, where for each \(i\), \(f_i : X \to X_i\). Then the weak topology \(\tau_\mathcal{R}\) on \(X\) determined by \(\mathcal{R}\) is the weakest topology \(\tau\) on \(X\) such that each \(f_i\) is continuous.

Evidently, a subbase for \(\tau_\mathcal{R}\) consists of all sets of the form \(f_i^{-1}(V_i)\) where \(V_i\) is open in \(\tau_i\). Thus, if the index set \(I\) is countable and each topology \(\tau_i\) is second countable, the weak topology is itself second countable. Since each space \((X_i, \tau_i)\) is Hausdorff, it is easy to check that the topology \(\tau_\mathcal{R}\) is Hausdorff if and only if the family \(\mathcal{R}\) separates points: whenever \(x_1 \neq x_2\), there exists \(i \in I\) such that \(f_i(x_1) \neq f_i(x_2)\).

**Theorem 5.9.** Suppose \(\mathcal{R} = \{f_i : i \in I\}\) induces a topology \(\tau_\mathcal{R}\) on a set \(X\). Then a net \(\{x_i\}\) in \(X\) is \(\tau_\mathcal{R}\)-convergent to \(x \in X\) if and only if \(\lim_{\lambda} f_i(x_i) = f_i(x)\) for each \(i \in I\).

**Proof.** If \(x \in \tau_\mathcal{R} - \lim x_i\) then for each \(i \in I\) we have \(f_i(x) = \lim_{\lambda} f_i(x_i)\) by the continuity of each \(f_i\). On the other hand, suppose we have \(f_i(x) = \lim_{\lambda} f_i(x_i)\) for each \(i \in I\). To show \(\tau_\mathcal{R}\)-convergence, let \(V\) be an arbitrary \(\tau_\mathcal{R}\)-open subset of \(X\). By the definition of \(\tau_\mathcal{R}\), there exist indices \(i_1, i_2, \ldots, i_n\) in \(I\) and open sets \(V_{i_1}, V_{i_2}, \ldots, V_{i_n}\) in the target space \(f_{i_1}, f_{i_2}, \ldots, f_{i_n}\) such that

\[
x \in \cap_{k=1}^n f_{i_k}^{-1}(V_{i_k}) \subset V.
\]
By continuity of \( f_k \) for \( k = 1, 2, \ldots, n \), there exists an index \( \lambda_0 \) in the underlying directed set for the net such that for all \( k \) and all \( \lambda \geq \lambda_0 \) we have \( f_k(x_\lambda) \in V_k \). As a result, for \( \lambda \geq \lambda_0 \) we have \( x_\lambda \in V \).

The following theorem says that the Hausdorff metric topology is a weak topology.

**THEOREM 5.10.** Let \((X,d)\) be a metric space. Then the Hausdorff metric topology on \( \text{cl}(X) \) is the weakest topology \( \tau \) on \( \text{cl}(X) \) such that for each \( B \in \text{cl}(X) \)

\[
\begin{align*}
e_d(B,\cdot) &: (\text{cl}(X),\tau) \rightarrow [0, +\infty], \\
e_d(\cdot,B) &: (\text{cl}(X),\tau) \rightarrow [0, +\infty], \\
D_d(B,\cdot) &: (\text{cl}(X),\tau) \rightarrow [0, +\infty),
\end{align*}
\]

are all \( \tau \)-continuous.

**Proof.** Let \( \tau_{\text{weak}} \) be the weak topology induced by the collection of all such functionals. Continuity of these functionals with respect to the Hausdorff metric topology follows from Theorem 5.7 and so \( \tau_{H_d} \supset \tau_{\text{weak}} \).

We now show that continuity of all functionals of the form \( e_d(B,\cdot) \) and \( e_d(\cdot,B) \) alone is enough to show that \( \tau_{\text{weak}} \) contains \( \tau_{H_d} \). Fix \( A_0 \in \text{cl}(X) \) and suppose \( A_0 = \tau_{\text{weak}} - \lim_{\lambda \downarrow} A_\lambda \). By Theorem 5.9 with \( B = A_0 \), we have \( \lim_{\lambda} e_d(A_0,A_\lambda) = e_d(A_0,A_0) = 0 \) and \( \lim_{\lambda} e_d(A_\lambda,A_0) = e_d(A_0,A_0) = 0 \). As a result

\[
\lim_{\lambda} H_d(A_\lambda,A_0) = \lim_{\lambda} \max\{e_d(A_0,A_\lambda) - e_d(A_\lambda,A_0)\} = 0
\]

and we obtain Hausdorff metric convergence of the net. This shows that \( \tau_{H_d} \subset \tau_{\text{weak}} \), completing the proof. \( \square \)

## 5.4 The Wijsman Topology

**Definition 5.11.** Let \((X,d)\) be a metric space. The Wijsman topology \( \tau_{\text{W}_d} \) on \( \text{cl}(X) \) is the weak topology determined by the family \( \{d(X,\cdot): x \in X\} \).

Since each distance functional is nonnegative, a subbase for the Wijsman topology consists of all sets of the form

\[
\begin{align*}
\{A \in \text{cl}(X) : d(x,A) < \alpha\} &: (x \in X, \alpha > 0), \\
\{A \in \text{cl}(X) : d(x,A) > \alpha\} &: (x \in X, \alpha > 0).
\end{align*}
\]

By Theorem 5.9 convergence of a net of closed sets \( \{A_\lambda\} \) to \( A \) in the Wijsman topology means that for each \( x \in X \) we have \( \lim_{\lambda} d(X,A_\lambda) = d(x,A) \). Thus, Wijsman convergence of a net of closed sets is the same as the pointwise convergence of the associated net of distance functionals. Put differently, the map \( A \rightarrow d(\cdot,A) \) is an embedding of \( (\text{cl}(X),\tau_{\text{W}_d}) \) into the space of continuous
functions \( C(X, \mathbb{R}) \), equipped with the topology of pointwise convergence. A useful characterization of Wijsman convergence is provided by the next theorem:

**Theorem 5.12.** Let \((X, d)\) be a metric space. Then the net \( \{A_i\}_{i \in \Lambda} \) in \( \text{cl}(X) \) is \( \tau_{w_2} \)-convergent to \( A \in \text{cl}(X) \) if and only if both of the following conditions are met:

1. Whenever \( A \) meets a nonempty open subset \( V \) of \( X \) then \( A_i \cap V \neq \emptyset \) eventually;
2. Whenever \( 0 < \varepsilon < \alpha \) and \( S_{\alpha}[x] \cap A_i = \emptyset \) eventually.

**Proof.** We claim that condition (1) is equivalent to the statement: \( \forall x \in X \), we have \( \limsup_i d(x, A_i) \leq d(x, A) \), whereas condition (2) is equivalent to the statement: \( \forall x \in X \), we have \( \liminf_i d(x, A_i) \geq d(x, A) \). We consider only (1), leaving (2) to the reader.

Suppose first that the \( \limsup \) inequality is satisfied for all \( x \) and \( V \) is an open subset of \( X \) such that \( A \cap V \neq \emptyset \). Choose \( a \in A \) and \( \varepsilon > 0 \) such that \( S_{\varepsilon}[a] \subset V \). Since \( d(a, A) = 0 \), eventually, \( d(a, A_i) < \varepsilon \) is satisfied and for all such \( \lambda \) we have \( A_i \cap S_{\varepsilon}[a] \neq \emptyset \). Condition (1) thus holds. Conversely, if (1) is satisfied, fix \( x \in X \), \( \delta > 0 \) and choose \( a \in A \) with \( d(x, a) < d(x, A) + \delta / 2 \). By assumption, \( A_i \cap S_{\delta/2}[a] \neq \emptyset \) for all large \( \lambda \) and for each such \( \lambda \) we have \( d(x, A_i) < d(x, a) + \delta / 2 < d(x, A) + \delta \). □

### 5.5 Hit-and-miss hyperspace topologies

A hit-and-miss hyperspace topology has a subbase that can be split into two halves. One kind of subbasic open set consists of all closed sets that hit each member of a particular family of open sets, whereas the other kind of subbasic open sets consists of all closed sets that avoid a particular kind of closed set. This simplest topology of this genre is the Vietoris topology, also called the finite topology or the exponential topology.

First, we need a little notation. If \( E \) is a subset of \( X \), we associate subsets \( E^- \) and \( E^+ \) of \( 2^X \) with \( E \) as follows:

\[
E^- = \{ A \in 2^X : A \cap E \neq \emptyset \}, \quad E^+ = \{ A \in 2^X : A \subset E \}.
\]

Thus, \( E^- \) consists of all closed sets that hit \( E \), whereas \( E^+ \) consists of all closed sets that miss \( E^c \).

**Definition 5.13.** The **Vietoris topology** \( \tau_V \) on \( 2^X \) has as a subbase all sets of the form \( V^- \), where \( V \) is open in \( X \) and all sets of the form \( W^+ \), where \( W \) is open in \( X \).

Admissibility of \( \tau_V \) is immediate. Let \( j : X \to 2^X \) be the canonical injection \( j(x) = \{ x \} \). The injection \( j \) is continuous because \( j^{-1}(V^-) = j^{-1}(V^+) = V \) and with respect to the relative topology on \( j(X) \), \( j \) is open because \( j(V) = V^- \cap j(X) \).

Although \( (V_1 \cap V_2)^- \neq V_1^- \cap V_2^- \), we do have \( (W_1 \cap W_2)^+ = W_1^+ \cap W_2^+ \). As a result, a typical basic open set determined by such subbasic open sets has the representation

\[
W^+ \cap \cap_{i=1}^k V_i^-,
\]
where $W, V_1, V_2, \ldots, V_k$ are open subsets of $X$.

An alternate subbase for $\tau_V$ restricted to $\text{cl}(X)$ consists of all sets of the form

$$[V_1, V_2, \ldots, V_k] = \{ A \in \text{cl}(X) : \forall i A \cap V_i \neq \emptyset \text{ and } A \subset \bigcup_{i=1}^k V_i \}.$$ 

Evidently, each $[V_1, V_2, \ldots, V_k]$ lies in $\tau_V$. On the other hand, for each open $V$ and $W$ we have, $V^- = [V, X]$ and $W^+ = [W]$. Thus, the topology generated by the sets of the form $[V_1, V_2, \ldots, V_k]$ does contain $\tau_V$. In fact, all sets of the form $[V_1, V_2, \ldots, V_k]$ actually form a base for the topology, as Vietoris first observed.

The Vietoris topology has been studied by many authors (Vietoris, Kelley, Kuratowski, McCoy, Beer and many others). It was proved that for metrizable $X$, the space $(2^X, \tau_V)$ is Hausdorff and if $X$ is connected (resp. compact) then $(\text{cl}(X), \tau_V)$ is connected (resp. compact).

From the perspective of multifunctions seen as set-valued functions, it has been useful to split the Vietoris topology into a lower half and an upper half.

Specifically, the **lower** (resp. **upper**) **Vietoris topology** is the topology on $2^X$ has as a subbase all sets of the form $V^-$ (resp. $V^+$) where $V$ runs over the open subsets of $X$. Now let $T$ be a topological space and let $F : T \rightarrow 2^X$ be a set-valued function. We declare $F$ **lower** (resp. **upper**) **semi-continuous** provided $F$ is continuous as a single valued function from $X$ to $2^X$, equipped with the lower (resp. upper) Vietoris topology.

Despite the attention given to the Vietoris topology by topologists, this hyperspace has had its detractors among analysts. The problem is this: the upper Vietoris topology is simply too strong. For example, in the plane, if $A = \{(x, y) : x = 0\}$ and if $A_n = \{(x, y) : x = 1/n\}$, one would like the sequence $\{A_n\}$ to converge to $A$. But $\{(x, y) : x = 0 \text{ or } y < 1/x^2\}^+$ is a neighbourhood of $A$ in the Vietoris topology that fails to contain any set $A_n$.

A response to this shortcoming of the Vietoris topology is the Fell topology $\tau_F$.

The Vietoris topology for the closed subsets of a metrizable space $X$ can be seen as a weak topology. Specifically, it can be seen as a topology determined by a family of distance functionals obtained by varying metrics as well as points.

**THEOREM 5.14.** Let $X$ be a metrizable space and let $\mathcal{D}$ denote the set of compatible metrics for $X$. Then the Vietoris topology $\tau_V$ on $\text{cl}(X)$ is the weak topology determined by the family of distance functionals

$$\{d(x, \cdot) : x \in X, d \in \mathcal{D}\}.$$ 

Thus, the Vietoris topology is the supremum in the lattice of hyperspace topologies of the Wijsman topologies determined by the compatible metrics for $X$, and a net $\{A_i\}$ in $\text{cl}(X)$ is $\tau_V$-convergent to $A \in \text{cl}(X)$ if and only if $\forall x \in X \forall d \in \mathcal{D}$, we have $d(x, A) = \lim_A d(x, A_i)$.

**Proof.** Denote the weak topology determined by the given family of distance functionals above by $\tau_{\text{weak}}$. We first show that $\tau_{\text{weak}} \subset \tau_V$.

Let $d \in \mathcal{D}$ and $x \in X$ be fixed. For each $\alpha > 0$, $\{B \in \text{cl}(C) : d(x, B) < \alpha\} = S_\alpha[x]^{-}$ is already in $\tau_V$. 

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On the other hand, if $A \in \{ B \in \text{cl}(X) : d(x, B) > \alpha \}$, then for some $\beta > \alpha$, we have $A \cap \overline{S}_{\beta}[x] = \emptyset$. Then

$$A \in (\overline{S}_{\beta}[x])^+ \subset \{ B \in \text{cl}(X) : d(x, B) > \alpha \}.$$  

This proves that each member of a subbase for $\tau_{W_d}$ belongs to $\tau_V$ and since $d$ was arbitrary we get $\tau_{\text{weak}} \subseteq \tau_V$.

For the reverse inclusion, Theorem 5.12 shows that each set of the form $V^-$, where $V$ is open in $X$, belongs to each Wijsman topology, and thus to $\tau_{\text{weak}}$. In the case that $W = X$, we have $W^+ = \text{cl}(X) \in \tau_{\text{weak}}$, and if $W = \emptyset$ then $W^+ = \emptyset \in \tau_{\text{weak}}$. Now let $W$ be an arbitrary proper open subset of $X$ and let $x_0 \in W^c$. Fix $A \in W^+$; we produce a compatible metric $\rho$ such that

$$A \in \{ B \in \text{cl}(X) : \rho(x_0, A) - \frac{1}{4} < \rho(x_0, B) \} \subset W^+.$$  

This would show that $W^+$ contains a $\tau_{\text{weak}}$-neighbourhood of each of its points.

To obtain $\rho$, let $d \in D$ be arbitrary. Since $A$ and $W^c$ are disjoint nonempty closed sets, we can find by Urysohn’s Lemma $\phi \in C(X, [0, 1])$ such that $\phi(A) = 0$ and $\phi(W^c) = 1$. Define $\rho : X \times X \to [0, 3/2]$ by

$$\rho(x, y) = \min\{\frac{1}{2}, d(x, y)\} + |\phi(x) - \phi(y)|.$$  

It is a routine exercise to verify that $\rho$ as defined is a metric equivalent to $d$.

Suppose that $\{ B \in \text{cl}(X) : \rho(x_0, A) - \frac{1}{4} < \rho(x_0, B) \} \subset W^+$. Then we can find $B \in \text{cl}(X)$ such that $\rho(x_0, A) - \frac{1}{4} < \rho(x_0, B)$, but at the same time $B \cap W^c \neq \emptyset$. Let $b$ be an arbitrary point of $B \cap W^c$. Since both $b$ and $x_0$ lie in $W^c$, we have $\rho(x_0, b) \leq 1/2$, so that

$$1 \leq \rho(x_0, A) - \rho(x_0, b) + \frac{1}{4} \leq \rho(x_0, B) + \frac{1}{4} \leq \frac{3}{4}.$$  

This is a contradiction and we conclude $W^+$ is indeed $\tau_{\text{weak}}$-open. We now may say that $\tau_V \subseteq \tau_{\text{weak}}$ and the asserted presentation of the Vietoris topology as a weak topology is established. The final statement is a consequence of Theorem 5.9.

\[\square\]

### 5.6 The Fell Topology

**Definition 5.15.** Let $X$ be a topological space. The Fell topology $\tau_F$ on $2^X$ has as a subbase all sets of the form $V^-$ where $V$ is open in $X$ and all sets of the form $(K^c)^+$ where $K$ is a compact subset of $X$.

Evidently, if $X$ is Hausdorff, the Fell topology is weaker than the Vietoris topology and it is easy to verify that this topology is admissible. Remarkably, the Fell topology is compact with no assumptions on $X$. To obtain this result, we use the Alexander subbase theorem.

**Theorem 5.16.** *Alexander subbase theorem.* Let $X$ be a topological space. Suppose each open cover of $X$ by a family of open sets chosen from a fixed subbase for the topology has a finite subcover. Then $X$ is compact.
Actually, to obtain compactness, we will show that each family of closed sets from the closed subbase associated with the standard open subbase that has empty intersection has a finite subfamily that has empty intersection.

**THEOREM 5.17.** Let \( X \) be an arbitrary topological space. Then \((2^X, \tau_F)\) is compact.

*Proof.* Suppose the family \( \{ (V^\lambda_\sigma) : \lambda \in \Lambda \} \cup \{ K^\sigma : \sigma \in \Sigma \} \) has empty intersection, where for each \( \lambda \), \( V_\lambda \) is open and for each \( \sigma \), \( K_\sigma \) is compact. Note each kind of set is represented, for if no set of the form \( K^\sigma \) appears, then \( \emptyset \in \cap \{(V^\lambda_\sigma) : \lambda \in \Lambda \} \) and if no set of the form \( (V^\lambda_\sigma)^+ \) appears, then \( X \in \cap \{ K^\sigma : \sigma \in \Sigma \} \). We claim that there exists \( \sigma_0 \in \Sigma \) with \( K^\sigma_0 \subset \cap V_\lambda \). Otherwise, pick for each index \( \sigma \in \Sigma \) a point \( x_\sigma \in K_\sigma \) that fails to lie in \( \cup V_\lambda \). Clearly, \( \text{cl} \{ x_\sigma : \sigma \in \Sigma \} \) lies in the intersection of the initial family, an impossibility. Thus, the asserted index exists. By compactness of \( K^\sigma_0 \), there exist indices \( \{ \Lambda_1, \Lambda_2, \ldots, \Lambda_n \} = \Lambda \) with \( \cup_{\iota=1}^n V_{\Lambda_\iota} \supset K^\sigma_0 \). From this, it is easy to check that the subfamily \( \{ K^{-\sigma_0} \} \cup \{ (V^\lambda_\iota)^+ : \iota \leq n \} \) has empty intersection. \( \square \)

As the following theorem shows, the Fell topology seems to work well only when \( X \) is locally compact.

**THEOREM 5.18.** Let \( X \) be a Hausdorff space. The following are equivalent:

(i) \( X \) is locally compact;

(ii) \( (2^X, \tau_F) \) is Hausdorff.

*Proof.* (i) \( \Rightarrow \) (ii). Let \( A \) and \( B \) be distinct closed sets. Without loss of generality, we may assume that \( A - B \neq \emptyset \). Let \( x \in A - B \) be arbitrary and let \( K \) be a compact neighbourhood of \( x \) disjoint from \( B \). Then \( \text{int} K^{-} \) and \( (K^{-})^+ \) are disjoint \( \tau_F \)-neighbourhoods of \( A \) and \( B \) respectively.

(ii) \( \Rightarrow \) (i). Since the hyperspace is Hausdorff, its points are closed, whence \( \text{cl}(X) \) is open in the hyperspace. By the last theorem, it follows that \( \text{cl}(X), \tau_F \) is a locally compact Hausdorff space. Now \( j(x) = \{ x \} \) actually embeds \( X \) as a closed subset of \((\text{cl}(X), \tau_F)\), for if \( B \in \text{cl}(X) \) contains two points \( b_1 \) and \( b_2 \) with disjoint neighbourhoods \( V_1 \) and \( V_2 \), we have \( j(X) \cap V_1^{-} \cap V_2^{-} = \emptyset \). Thus \( X \) is a closed subset of a locally compact space and therefore itself locally compact. \( \square \)

**Remark 5.19.** Beer (in Topologies on Closed and Closed Convex Sets) shows that if \( X \) is a Hausdorff space then \((\text{cl}(X), \tau_F)\) is metrizable if and only if \( X \) is locally compact and second countable. Both of these properties are equivalent with \((2^X, \tau_F)\) being a compact metrizable space.

### 5.7 Functions, multifunctions, metafunctions

If \( X, Y \) are topological spaces, a multifunction \( F : X \rightharpoonup Y \) with closed values can be perceived as a function \( F : X \to \text{cl}(Y) \). But sometimes, as the following Example shows, we could be interested in investigating multifunctions from \( X \) to \( \text{cl}(Y) \). (A more thorough look at these special multifunctions - that are called metafunctions - will be taken in a special chapter.)
Example 5.20. Let $\lambda$ be the Lebesgue measure defined on $\mathbb{R}$. We can define a multifunctions $F : [0, +\infty) \rightharpoonup C(\mathbb{R})$ in the following way:

$$
\forall t \in [0, +\infty) F(t) = \{ A : \lambda(A) = t \}.
$$

This multifunction can be seen as an "inverse mapping" to $\lambda$ defined on $C(\mathbb{R})$.

Exercises

Exercise 1 If $(X, T)$ is defined by $X = \mathbb{R}$, $T = \{\emptyset\} \cup \{ \mathbb{R} \setminus A : A \text{ is finite} \}$, what can you say about the Vietoris topology and the Fell topology on $2^X$?

Exercise 2 If $X, Y$ are topological spaces, $F : X \rightharpoonup Y$ a multifunction with closed values and $\tau_{VU}$ ($\tau_{VL}$) are the upper Vietoris (lower Vietoris) topologies on $\text{cl}(Y)$, define "$F$ is l.s.c., $F$ is u.s.c." in terms of a continuity properties for the function $F : X \to \text{cl}(Y)$.

Exercise 3 Find a function $f : \mathbb{R} \to \text{cl}(\mathbb{R}^2)$ that is continuous with respect to the Fell topology on $\text{cl}(\mathbb{R}^2)$ but it is not continuous with respect to the Vietoris topology on $\text{cl}(\mathbb{R}^2)$. 
