Chapter 4

Real-valued functions

The subject as well as the methods of study of a class of mappings depend crucially on structures of the sets which the definition domains and values are chosen from. The mathematical analysis is interested especially in the mappings from $\mathbb{R}^m$ to $\mathbb{R}^k$ for $m, k \in \mathbb{N}$. These mappings appear frequently in many fields of applications of mathematics. Namely such the mappings are used in physics, technical sciences, natural sciences, economy and social sciences (like psychology and sociology). This is the reason why we will be interested in various kinds of this class of mappings and in their properties.

4.1 Functions and operations with them

Mostly, we will be interested in the following class of mappings:

**Definition 4.1.1.** Let $A, B$ be sets. The mapping $f : A \rightarrow \mathbb{R}$ is called the real-valued function (or simply function). The mapping $g : B \rightarrow \mathbb{R}^k$, $k \in \mathbb{N}$ is called the vector function and the mapping $h : (A \subset \mathbb{R}^m) \rightarrow B$, $m \in \mathbb{N}$ is called the abstract function.

Especially, the mapping $f : (A \subset \mathbb{R}^m) \rightarrow \mathbb{R}$, $m \in \mathbb{N}$ is called the real-valued function of $m$ real variables. (For $m = 1$ this simplifies to $f : (A \subset \mathbb{R}) \rightarrow \mathbb{R}$).

The examples of such the mappings can be found on various figures in previous chapters. An abstract function $g : (T \subset \mathbb{R}) \rightarrow \mathbb{R}^2$ was used when defining the curve in Definition 2.4.1 and in Example 2.4.1.

The fact that we have defined operations of addition, subtraction, multiplication and division allows us to introduce all these operations also for the functions and build up with help of these operations new kinds of mappings.

**Definition 4.1.2.** Let $A$ be an arbitrary nonempty set and let $f, g : A \rightarrow \mathbb{R}$. We define:
4.1 Functions and operations with them

1. Addition of the functions $f$ and $g$ by

$$f + g : A \to \mathbb{R}; \quad (f + g)(x) = f(x) + g(x), \ x \in A.$$ 

2. Subtraction of the functions $f$ and $g$ by

$$f - g : A \to \mathbb{R}; \quad (f - g)(x) = f(x) - g(x), \ x \in A.$$ 

3. Product of the functions $f$ and $g$ by

$$f \cdot g : A \to \mathbb{R}; \quad (f \cdot g)(x) = f(x) \cdot g(x), \ x \in A.$$ 

4. If there is such a subset $M$ in $A$ that $g(x) \neq 0$ for $x \in M$ then we define fraction of the functions $f$ and $g$ by

$$f/g : M \to \mathbb{R}; \quad (f/g)(x) = f(x)/g(x), \ x \in M.$$ 

As a special case we obtain multiplication of a real-valued function by a real number.

We see, by Definition 4.1.2, that we can define, over any set of mappings $f : A \to B$, the same operations those are defined on the set $B$. The following theorem holds true:

**THEOREM 4.1.1.** Let $A$ be an arbitrary set and let $F$ be the set of all real-valued functions $f : A \to \mathbb{R}$. Then the structure:

a) $(F; +, \cdot, =)$ is the commutative and unitary ring;

b) $(F, \mathbb{R}; +, \cdot, =)$ is the vector space over the field of real numbers $\mathbb{R}$.

**Proof.** It is sufficient to verify the properties of the ring from Definition 1.6.4 and properties of the vector space from Definition 1.6.5, respectively.

Let us remind that the neutral element with respect to addition is given by the function: $e_1 : x \mapsto 0, \ x \in A$. The inverse element (with respect to addition) - also called *additive inverse* to an element $f$ is given by the function: $\bar{f} : x \mapsto -f(x), \ x \in A$. The neutral element with respect to multiplication is given by the function: $e_2 : x \mapsto 1, \ x \in A$.

The proof can be concluded by the fact that the set of all real numbers (that is the set of values of considered mappings) is both the commutative and unitary ring as well as the vector space (over the field $\mathbb{R}$).

We see that the structure $(F; +, \cdot, =)$ is not a domain or a field since there is no an multiplicative inverse to any $f$ different from $e_1$.

There is the following generalization of the statement b) of Theorem 4.1.1:
4.2 Monotonic functions

**Theorem 4.1.2.** Let \( F \) be a set of all mappings \( f : A \to \mathbb{R}^k, \ k \in \mathbb{N} \). The mathematical structure \( (F, \mathbb{R}, +, \cdot, =) \) (with the operations defined on the set of ordered \( k \)-tuples from Theorem 1.6.4) is the vector space over the field of real numbers \( \mathbb{R} \).

**Proof.** The proof of the theorem consists of simple verification of the axioms of a vector space. \( \square \)

**Problems**

1. Prove in details Theorem 4.1.2.

2. Let the mappings: \( f, g : A \to \mathbb{R} \) be bounded in \( \mathbb{R} \). Find out whether the real-valued functions \( f \pm g, \ f \cdot g \) and if exists then \( f \circ g \) (all of them acting from \( A \to \mathbb{R} \)) are bounded in \( \mathbb{R} \).

3. Is the following function:

\[
\begin{align*}
f : x & \mapsto \frac{x + 1}{x^2 - 1}, \ x \in \mathbb{R} \setminus \{-1, 1\},
\end{align*}
\]

bounded in \( \mathbb{R} \)?

4. Find the definition domains of the following functions

\[
\begin{align*}
f : x & \mapsto \sqrt{3x - x^2}, & g : x & \mapsto \frac{1}{\sqrt{x^2 + 5x + 4}} + \frac{1}{x^2 - 2}, \\
h : x & \mapsto \frac{1}{1 + |1 - x|} + \frac{1}{1 + x^2}.
\end{align*}
\]

**Answers**

2. The functions are bounded in \( \mathbb{R} \).

3. The function is not bounded in \( \mathbb{R} \).

4. 

\[
D(f) = (-\infty, -\sqrt{3}) \cup [0, \sqrt{3}]; \ D(g) = (-\infty, -4) \cup (\sqrt{2}, +\infty); \ D(h) = \mathbb{R}.
\]

4.2 Monotonic functions

We will discuss some special kinds of real-valued functions of one real variable in this section.
4.2 Monotonic functions

4.2.1 Even and odd functions, periodic functions

Definition 4.2.1. Let $f : (A \subset \mathbb{R}) \to \mathbb{R}$. We say that the function $f$ is

a) even iff the following holds true

$$x \in A \Rightarrow [-x \in A \& f(x) = f(-x)],$$

b) odd iff the following holds true

$$x \in A \Rightarrow [-x \in A \& f(x) = -f(-x)],$$

c) $p$-periodic ($0 \neq p \in \mathbb{R}$) iff the following holds true

$$x \in A \Rightarrow [x \pm p \in A \& f(x \pm p) = f(x)].$$

Such a number $p$ is then called the period of the function $f$.

4.2.2 Monotonic functions

Starting with the fact that the set of all real numbers $\mathbb{R}$ is ordered by the relation of natural ordering, we can compare by the same ordering also the values of real-valued functions, especially we can say that a function $f : (A \subset \mathbb{R}) \to \mathbb{R}$ is non-negative ($f \geq 0$) iff for all $x$ in $A$ we have $f(x) \geq 0$. And by this we can define new and important class of monotonic functions

Definition 4.2.2. Let $f : (A \subset \mathbb{R}) \to \mathbb{R}$ and let $M \subset A$. We say that the function $f$ is

a) increasing on $M$ iff

$$x_1, x_2 \in M, \ x_1 \neq x_2 \Rightarrow [f(x_1) - f(x_2)](x_1 - x_2) > 0$$

b) decreasing on $M$ iff

$$x_1, x_2 \in M, \ x_1 \neq x_2 \Rightarrow [f(x_1) - f(x_2)](x_1 - x_2) < 0$$

c) nondecreasing on $M$ iff

$$x_1, x_2 \in M, \ x_1 \neq x_2 \Rightarrow [f(x_1) - f(x_2)](x_1 - x_2) \geq 0$$

d) nonincreasing on $M$ iff

$$x_1, x_2 \in M, \ x_1 \neq x_2 \Rightarrow [f(x_1) - f(x_2)](x_1 - x_2) \leq 0$$

28
4.2 Monotonic functions

- If a function $f$ is of one of above kinds, then we call it monotonic on $M$.
- Is a function $f$ is either increasing or decreasing on $M$ we say that it is strictly monotonic on $M$.

**Example 4.2.1.** Are the following functions monotonic or periodic?

a) 
$$ f : x \mapsto \frac{1}{x}, \ x \in \mathbb{R} \setminus \{0\}; $$

b) 
$$ \text{sgn} : x \mapsto \begin{cases} 
1 & \text{for } x > 0, \\
0 & \text{for } x = 0, \ x \in \mathbb{R}; \\
-1 & \text{for } x < 0,
\end{cases} $$

c) 
$$ \chi : x \mapsto \begin{cases} 
1 & \text{for } x \in \mathbb{Q}, \\
0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \ x \in \mathbb{R},
\end{cases} $$

(the so-called Dirichlet function);

d) 
$$ g : x \mapsto [x], \ x \in \mathbb{R}; $$

(the so-called floor function, or, when defined over the set of positive numbers, also called integral part of a number).

These functions are plotted on Fig. 4.1.

**Solution:** We will discuss all the functions, using also Fig. 4.1.

- Let $x_1 \neq x_2$ be from $D(f)$, then

$$ \left( \frac{1}{x_1} - \frac{1}{x_2} \right) (x_1 - x_2) = -\frac{(x_1 - x_2)^2}{x_1 x_2}. $$

This means that the function $f$ is decreasing on $(-\infty, 0)$ and also on $(0, \infty)$, however, it is not decreasing nor increasing on $D(f)$. The function $f$ is obviously not periodic.

- The functions signum and floor are nondecreasing on $\mathbb{R}$. In fact, for any $x_1, x_2 \geq 0$ or $x_1, x_2 \leq 0$ that are different ($x_1 \neq x_2$) we have

$$ (\text{sgn}(x_1) - \text{sgn}(x_2))(x_1 - x_2) \geq 0. \quad (\star) $$

For mutually different $x_1, x_2$ such that $x_1 \geq 0$ and $x_2 \leq 0$ or $x_1 \leq 0$ and $x_2 \geq 0$ we have that the left-hand side of $(\star)$ is given by

$$ 2(x_1 - x_2) \geq 0, $$
or by

\[-2(x_1 - x_2) \geq 0.\]

Similarly, we can proof the same about the floor function.

c) The Dirichlet function is not monotonic on $\mathbb{R}$. It is, at the same time, nondecreasing and nonincreasing on (separately) $\mathbb{Q}$ and on $\mathbb{R} \setminus \mathbb{Q}$. Moreover, it is $p$-periodic with arbitrary rational period $p \in \mathbb{Q}$. In fact, for any rational numbers $x, p$, the numbers $x \pm p$ are also rational, and for any irrational $x$ and rational $p$ the numbers $x \pm p$ are also irrational. Therefore, for any real $x$ and for any rational $p$ the following equality holds true:

\[\chi(x \pm p) = \chi(x).\]

\[\chi(x \pm p) = \chi(x).\]

**Figure 4.1:**

4.2.3 Properties of (strictly) monotonic functions

In this subsection we will be interested in relation between monotonic functions and injective functions. The following theorem holds true

**Theorem 4.2.1.** If a function $f : (A \subset \mathbb{R}) \to \mathbb{R}$ is strictly monotonic on $A$ then it is injective.
4.2 Monotonic functions

Proof. Let \( f \) be increasing on \( A \). Then for all \( x_1, x_2 \in A \) such that \( x_1 \neq x_2 \) we have

\[
[f(x_1) - f(x_2)](x_1 - x_2) > 0.
\]

Thus, \( f(x_1) \neq f(x_2) \). For a decreasing function, the proof can be done in the same way. \( \square \)

We can easily see that the implication in Theorem 4.2.1 cannot be inverted, since there are functions that are injective but are not monotonic. As an example of such a function we have:

\[
f : x \mapsto \frac{1}{x}, \ x \in \mathbb{R} \setminus \{0\}.
\]

THEOREM 4.2.2. Let a function \( f : (A \subset \mathbb{R}) \to \mathbb{R} \) be increasing (decreasing) on \( A \). Then the inverse function \( f^{-1} : f(A) \to A \) is increasing (decreasing) on \( f(A) \).

Proof. We will prove the theorem for a decreasing function. Theorem 4.2.1 ensures the existence of the inverse function \( f^{-1} \) (see also Theorem 2.2.1). Now, let \( y_1, y_2 \in f(A) \) be such that \( y_1 \neq y_2 \). The there are \( x_1, x_2 \in A \) such that

\[
y_1 = f(x_1), \ y_2 = f(x_2),
\]

and \( x_1 \neq x_2 \). (Otherwise, we would have \( y_1 = y_2 \).) Thus,

\[
[f^{-1}(y_1) - f^{-1}(y_2)](y_1 - y_2) = (x_1 - x_2)[f(x_1) - f(x_2)] < 0,
\]

and this inequality verifies that \( f^{-1} \) is decreasing on \( f(A) \). \( \square \)

All the notions discussed above for real-valued functions of one real variable can be applied also for the sequences of real numbers. Especially, the following theorem holds true for the sequences of real numbers:

THEOREM 4.2.3. A sequence of real numbers \( (a_n)_{n \in \mathbb{N}} \) is increasing (decreasing) iff

\[
\forall n \in \mathbb{N} : a_n < a_{n+1} \quad (a_n > a_{n+1}).
\]

Example 4.2.2. Let the function \( g : (A \subset \mathbb{R}) \to (B \subset \mathbb{R}) \) be increasing on \( A \) and let the function \( f : (E \supset B) \to \mathbb{R} \) be increasing on \( E \). Then the composite function \( f \circ g : A \to \mathbb{R} \) is increasing on \( A \). Prove this!

Solution: For any two \( x_1, x_2 \in A \) such that \( x_1 < x_2 \) we have \( g(x_1) < g(x_2) \) and also \( f[g(x_1)] < f[g(x_2)] \). Therefore, we have

\[
\{f[g(x_1)] - f[g(x_2)]\}(x_1 - x_2) > 0.
\]

The same inequality holds true also for \( x_1 > x_2 \). So, we can conclude that \( f \circ g \) is increasing on \( A \).
4.2.4 Bounded functions

The following simple statement holds true for the bounded functions (see definition 2.1.4).

**Theorem 4.2.4.** A real-valued function \( f : A \rightarrow \mathbb{R} \) is bounded (in \( \mathbb{R} \)) on a set \( M \subset A \) if and only if
\[
\exists k \in \mathbb{R} \quad \forall x \in M : |f(x)| \leq k.
\]

*Proof.* We split the proof, as usually, into two implications:
1. By using definition 2.1.4 we have that the values \( H(f|_M) \) of the restriction \( f|_M \) is a bounded set in \( \mathbb{R} \). Thus, we know that there are real numbers \( k_1 \leq k_2 \) such that
\[
\forall y \in H(f|_M) : k_1 \leq y \leq k_2.
\]
   For any \( y \in H(f|_M) \) we have an \( x \in M \) such that \( y = f(x) \). Then we can rewrite the last statement into the form
\[
\forall x \in M : -k \leq f(x) \leq k \iff |f(x)| \leq k,
\]
   where \( k = \max\{|k_1|, |k_2|\} \). This concludes the proof of the necessary condition.
2. The inequality \( |f(x)| \leq k, \ x \in M \) means that for \( x \in M \) we have \(-k \leq f(x) \leq k\). So, we have that the lower (upper) bound of the set \( H(f|_M) \) is \(-k \ (k)\). \(\square\)

4.2.5 Supremum of a function

By using the previous knowledge we can formulate and prove the following theorem on supremum of a function

**Theorem 4.2.5.** Let a real-valued function \( f : A \rightarrow \mathbb{R} \) be bounded from above in \( \mathbb{R} \). Then the following statements are mutually equivalent:

- \[
\sup f = b \in \mathbb{R},
\]

- \[
\forall x \in A : f(x) \leq b,
\]

- Let \( y \in \mathbb{R} \) be such that
  \[
  (\forall x \in A : f(x) \leq y) \Rightarrow b \leq y
  \]
  \[
  \forall x \in A : f(x) \leq b,
  \]

- \[
(\forall y \in \mathbb{R} : y < b) \exists x_0 \in A : y < f(x_0)
  \]
  \[
  \forall x \in A : f(x) \leq b,
  \]
4.2 Monotonic functions

\[ \forall \epsilon > 0 \exists x_0 \in A : b - \epsilon < f(x_0). \]

**Proof.** 1. (1) \(\Leftrightarrow\) (2) follows directly from Definition 1.5.4 of the supremum of a ordered set \(B = H(f)\) and from Definition 2.1.4 of the supremum of a mapping \(f\).

2. (2) \(\Leftrightarrow\) (3) follows from Theorem 1.5.3 for \(B = h(f)\).

3. (3) \(\Leftrightarrow\) (4) can be obtained from Theorem 3.1.1.

4. (4) \(\Leftrightarrow\) (1) follows directly from Theorem 3.1.1. \(\square\)

An analogical theorem holds true also in \(\tilde{\mathbb{R}}\).

**Example 4.2.3.** Let the functions \(f, g : A \to \mathbb{R}\) be bounded from above in \(\mathbb{R}\). Then

\[
\sup(f + g) \leq \sup f + \sup g, \quad \text{(A)}
\]

\[
\sup(\lambda f) = \lambda \sup f, \quad \lambda \in \mathbb{R}^+, \quad \text{(B)}
\]

\[
|\sup f - \sup g| \leq \sup |f - g|. \quad \text{(C)}
\]

Prove these relations!

**Solution:** 1. Since the function \(f + g\) is bounded from above, all the suprema in Eq. (A) do exist. It follows from the first property of the supremum that for all \(x \in A\) we have

\[
f(x) \leq \sup f, \quad g(x) \leq \sup g,
\]

therefore

\[
f(x) + g(x) \leq \sup f + \sup g.
\]

The real number \(\sup f + \sup g\) is an above bound of the function \(f + g\), and from the second property of the supremum we have

\[
\sup(f + g) \leq \sup f + \sup g.
\]

2. For all \(x \in A\) and \(\lambda > 0\) we have \(f(x) \leq \sup f \Rightarrow \lambda f(x) \leq \lambda \sup f \Rightarrow \sup \lambda f \leq \lambda \sup f\).

From the other side for \(x \in A\) we have

\[
\lambda f(x) \leq \sup \lambda f \Rightarrow \frac{1}{\lambda} \sup \lambda f \Rightarrow \sup f \leq \frac{1}{\lambda} \sup \lambda f \Rightarrow \lambda \sup f \leq \sup \lambda f.
\]

So, the formula (B) holds true.

3. Property (A) implies

\[
\sup f = \sup(f - g + g) \leq \sup(f - g) + \sup g \leq \sup |f - g| + \sup g.
\]

The inequality \(\sup(f - g) \leq \sup |f - g|\) follows from that: \(\forall x \in A : f(x) - g(x) \leq |f(x) - g(x)|\).

Furthermore, we have

\[
\sup f - \sup g \leq \sup |f - g|.
\]

Similarly,

\[
\sup g - \sup f \leq \sup |g - f| = \sup |f - g|.
\]

The last two inequalities imply the required formula (C).
Problems

1. Show that for \( n \in \mathbb{N} \) the function \( c \mapsto x^n \) is even for even \( n \) and is odd for odd \( n \).

2. Find a function that is at the same time even and odd.

3. Can a periodic function be monotone? What is the period of such a function equal to?

4. Can a function be, at the same time, increasing and even (decreasing and odd)?

5. Explain what are the difficulties when trying to define monotonicity of a function from \( \mathbb{R}^m \) with \( m \geq 2 \) (of a vector function from \( \mathbb{R}^m \) to \( \mathbb{R}^k \)). What are the difficulties when trying to define the boundedness of a vector function of a vector argument?

6. Prove Theorem 4.2.2 for an increasing function.

7. Prove Theorem 4.2.3 and formulate an analogical theorems for nondecreasing function and nonincreasing function, respectively.

8. Formulate and prove statements analogical to those from Theorem 4.2.5 concerning the infimum of a function.

9. Prove the following. Let the functions \( f, g : A \to \mathbb{R} \) be bounded from below in \( \mathbb{R} \). Then:

\[
\inf(f + g) \geq \inf f + \inf g, \quad \sup(\lambda f) = \lambda \inf f, \ \lambda < 0.
\]

10. Examine the function: \( f : x \mapsto x - \lfloor x \rfloor := \{x\} \) (called also fractional part of \( x \)) for periodicity, parity and monotonicity.

11. Examine the composite function \( f \circ g \) (when we know that (1) both functions \( f \) and \( g \) are decreasing or (2) one of them is increasing and the second one decreasing) for monotonicity.

Answers

2 \( f : x \mapsto 0, x \in (-a, a), a \in \mathbb{R} \).

3 Yes. Any real number can be the period of such a function.

4 No (no).

10 The function \( f \) is 1-periodic and is not even nor odd.
4.3 Polynomials and rational functions

In this section we will define and work with very often used kinds of functions - polynomials and their ratios, i.e. rational functions. The values of these functions are defined very simply by operations of addition, subtraction multiplication and constant, especially, only integer-number exponents of variable appear in polynomials. These mappings are often used in the theory of approximation of functions and we will meet them again in chapter 10 devoted to integration.

4.3.1 Polynomials

For \( x \in \mathbb{C} \) and \( n \in \mathbb{N} \) we have defined the \( n \)-th power of \( x \) by

\[
x^n = x \cdot x \cdots x, \quad n \text{-times}
\]

**Definition 4.3.1.** The mapping \( P_{n,m} : \mathbb{C}^m \to \mathbb{C} \) (alternatively, \( P_{n,m} : \mathbb{R}^m \to \mathbb{R} \)) defined by

\[
x = (x_1, x_2, \ldots, x_m) \mapsto \sum_{0 \leq i_1 + \cdots + i_m \leq n} a_{i_1 \ldots i_m} x_1^{i_1} \cdots x_m^{i_m}, \quad x \in \mathbb{C}^m
\]

is called the *polynomial of \( n \)-th degree (order) (of many complex (real) variables)*, where the coefficients \( a_{i_1 \ldots i_m} \in \mathbb{C} \) and at least one of the coefficients such that \( i_1 + \cdots + i_m = n \) is different from zero.

The degree of a polynomial \( P \) is denoted by \( \deg P \). For example, the real-valued function

\[
P_{1,m} : \quad x = (x_1, \ldots, x_m) \mapsto a_1 x_1 + \cdots + a_m x_m + b, \quad x \in \mathbb{R}^m
\]

is the polynomial of the first degree for \( a_i \in \mathbb{R}, \quad i = 1, 2, \ldots, m, \quad b \in \mathbb{R} \) if at least one of the coefficients \( a \) is different from zero. (This polynomial, in case \( b = 0 \), is a linear function (mapping)). The mapping

\[
P_{n,1} =: P : \quad x \mapsto a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad x \in \mathbb{C}
\]

for \( a_i \in \mathbb{C}, \quad i = 0, 1, 2, \ldots, n \in \mathbb{N} \) and \( a_n \neq 0 \) is the polynomial of \( n \)-th degree (of one complex variable).

The function

\[
P_{2,2} : \quad (x, y) \mapsto xy + x + y + 3, \quad (x, y) \in \mathbb{R}^2
\]

is the polynomial of the degree two.

The constant nonzero function

\[
x \mapsto a \in \mathbb{R} \setminus \{0\}, \quad x \in \mathbb{R},
\]

\[
ax^{m-1}, \quad m \in \mathbb{N}, \quad a \in \mathbb{R} \setminus \{0\}
\]
is the polynomial of degree zero.
The function
\[ x \mapsto 0, \ x \in \mathbb{C} \]
is called the zero polynomial.
The set of all polynomials of form (1) forms the infinitely dimensional vector space over \( \mathbb{R} \) which basis is \( \{ 1, x, x^2, x^3, \ldots \} \) (see section 1.6).

**Roots of the polynomials**

In the next text we will be interested in polynomials of the form (1).

**Definition 4.3.2.** The complex number \( z \) is called the root of the polynomial (1) iff the equality \( P(z) = 0 \) holds true.

It is obvious that the polynomial of degree zero has no root. The base of study the roots of the polynomials is the so-called fundamental theorem of algebra:

**THEOREM 4.3.1.** A polynomial of the form (1) has at least one root (from \( \mathbb{C} \)).

We have not enough of knowledge at this time to prove this theorem. We refer the reader to the literature, namely the book [11] (pages 221-223) or [8] (pages 147-155).

**THEOREM 4.3.2.** Let \( z \in \mathbb{C} \) be a root of the polynomial \( P \) (1) of degree \( n \) equal or greater than 1. Then there is a polynomial \( Q \) of degree \( n-1 \) such that
\[ P(x) = Q(x)(x - z), \ \forall x \in \mathbb{C}. \]

**Proof.** Since \( P(z) = 0 \) we can write
\[
P(x) = P(x) - P(z) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 - (a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0) = \\
= a_n (x^n - z^n) + a_{n-1} (x^{n-1} - z^{n-1}) + \cdots + a_1 (x - z) = \\
= (x - z) [a_n (x^{n-1} + z x^{n-1} + \cdots + z^{n-2} x + z^{n-1}) + \cdots + a_1] = \\
= (x - z) Q(x),
\]
where \( Q : x \mapsto a_n (x^{n-1} + z x^{n-1} + \cdots + z^{n-2} x + z^{n-1}) + \cdots + a_1 \) is really the polynomial of degree \( n - 1 \) since we have supposed \( a_n \neq 0 \). \( \square \)

**Definition 4.3.3.** We say that the complex number \( z \) is the root of multiplicity \( k \) of the polynomial \( P \) (1) iff the following equality holds true:
\[ P(x) = (x - z)^k Q(x), \forall x \in \mathbb{C} \quad \text{and} \quad Q(z) \neq 0. \]

**THEOREM 4.3.3.** Let \( P \) be a polynomial of the form (1) of degree \( n \geq 1 \). Then \( P \) has just \( n \) roots when counting any root with its multiplicity.
4.3 Polynomials and rational functions

Proof. By induction. If the degree of $P$ is 1 then the single root can be easily computed. So, the statement holds true for $n = 1$. Let the theorem holds true for degree $n$. Let $P$ has degree $n + 1$. Then Theorem 4.3.1 implies existence of a root $z \in \mathbb{C}$ of $P$ such that with respect to Theorem 4.3.2 we can write

$$P(x) = (x - z)Q(x), \ x \in \mathbb{C},$$

where $Q$ is a polynomial of degree $n$. By assumption, $Q$ satisfies theorem, thus $P$ satisfies it, too.

Corollary 4.3.1. Any polynomial $P : x \mapsto a_n x^n + \cdots + a_1 x + a_0, \ x \in \mathbb{C}$ of degree $n \geq 1$ can be written as

$$P(x) = a_n (x - z_1)^{k_1} (x - z_2)^{k_2} \cdots (x - z_i)^{k_i}, \ x \in \mathbb{C},$$

where $z_1, \ldots, z_i$ are the roots of $P$ and $k_1, \ldots, k_i$ are their multiplicities. Moreover, we have $k_1 + \cdots + k_i = n$.

Theorem 4.3.4. Let $P, Q$ be polynomials of one variable. The equality $P = Q$ holds true iff all the corresponding coefficients of both polynomials (i.e. coefficients standing by equal powers of the variable) are equal.

Proof. 1. Let $P = Q$ and

$$P : x \mapsto a_n x^n + \cdots + a_1 x + a_0, \ x \in \mathbb{C}, \ Q : x \mapsto b_n x^n + \cdots + b_1 x + b_0, \ x \in \mathbb{C}.$$

We can suppose without lost of generality that $n \geq m$. If the corresponding coefficients are not mutually equal then we can denote by $i$ the maximal index with the property $a_i \neq b_i$. Then

$$P - Q : x \mapsto (a_i - b_i) x^i + \cdots + (a_1 - b_1) x + (a_0 - b_0), \ x \in \mathbb{C}.$$

If $i = 0$ then for all complex $x$ we have $P(x) \neq Q(x)$. If $i > 0$ then the polynomial $P - Q$ has positive degree and with respect to Theorem 4.3.3 has only finite number of roots - namely $i$. Thus, there exists $x_0 \in \mathbb{C}$ such that $P(x_0) \neq Q(x_0)$. In both cases, we are in contradiction with the assumption.

2. If all the corresponding coefficients of the polynomials $P$ and $Q$ are mutually equal then for all the complex numbers $x$ we have $P(x) = Q(x)$ and this means (by Theorem 2.1.1) that $P = Q$.

Theorem 4.3.5. Let $P$ be a polynomial of the form (1) with real coefficients. If a complex number $z$ is a root of $P$ then also the complex conjugate $\bar{z}$ is the root of $P$. The multiplicities of the roots $z$ and $\bar{z}$ are equal.
4.3 Polynomials and rational functions

Proof. If the root $z$ of $P$ is real then the theorem holds true trivially. We will suppose $z \in \mathbb{C} \setminus \mathbb{R}$ and split the proof into two steps.

1. Let $P(z) = 0$. Since the coefficients are real, we can write

$$P(z) = a_n(z)^n + \cdots + a_1z + a_0 = a_n(z^n) + \cdots + a_1z + a_0 =$$

$$= a_nz^n + \cdots + a_1z + a_0 = 0.$$

2. Now, we will prove that if the root $z$ of $P$ has the multiplicity $k$ then $\bar{z}$ has also the multiplicity $k$. Let us suppose that the multiplicity of $z$ is $k$ and the multiplicity of $\bar{z}$ is $l$. Furthermore, we will suppose that $k > l$. Then we can write (by Definition 4.3.3):

$$\forall x \in \mathbb{C} : P(x) = (x - z)^k(x - \bar{z})^l Q(x)$$

where $Q(\bar{z}) \neq 0$ and $Q(z) = 0$. However, this is in contradiction with the result of the first part of this proof. The same can be derived starting with $k < l$. Thus, it must be $k = l$. □

Corollary 4.3.2. The value $P(x)$ of any polynomial $P$:

$$P : x \mapsto a_nx^n + \cdots + a_1x + a_0, x \in \mathbb{C},$$

with real coefficients, can be written in the following form:

$$P(x) = a_n(x - x_1)^{k_1} \cdots (x - x_i)^{k_i} (x^2 + p_1x + q_1)^{l_1} \cdots (x^2 + p_jx + q_j)^{l_j}, \quad (2)$$

where $x_1, \ldots, x_i$ are all the real roots of $P$ (with corresponding multiplicities $k_1, \ldots, k_i$). The numbers $p_1, q_1, \ldots, p_j, q_j$ are real and the polynomials $x \mapsto x^2 + p_kx + q_k, x \in \mathbb{C}, k = 1, 2, \ldots, j$ have no real roots and $k + 1 + \cdots + k + i + 2(l_1 + \cdots + l_j) = n$.

Proof. The proof follows Corollary 4.3.1 and from that for any $z \in \mathbb{C}$ we have

$$(x - z)(x - \bar{z}) = x^2 + px + q,$$

where the numbers $p$ and $q$ are real. □

The equality (2) is called the factorization of a polynomial.

Example 4.3.1. Perform the factorization (2) of the polynomial

$$P : x \mapsto x^6 + x^4 - x^2 - 1, \quad x \in \mathbb{R}.$$  

Solution: One can easily see that 1 is a root of this polynomial. Subsequently, we obtain:

$$P(x) = (x - 1)(x^5 + x^4 + 2x^3 + x + 1) = (x - 1)(x + 1)(x^4 + 2x^2 + 1) =$$

$$= (x - 1)(x + 1)(x^2 + 1)^2.$$
4.3 Polynomials and rational functions

4.3.2 Rational functions

In the rest of this section, we will be interested in basic properties of the rational functions (ratios of the polynomials), and, especially, their factorization - this procedure will play an important role in integration of the rational functions.

**Definition 4.3.4.** Let \( P, Q : \mathbb{C}^m \to \mathbb{C} \) be two polynomials of degree \( n \) with real coefficients (see Definition 4.3.1). Let \( Q(x) \neq 0 \) for \( x \in M \subset \mathbb{C} \). Then the fraction:

\[
P / Q : M \to \mathbb{C}
\]

is called the rational function (of \( m \) complex variables).

For example,

\[
S : (x, y, z) \mapsto \frac{xyz + x + y - 1}{x^2 + y^2 - 1}, \ (x, y, z) \in M = \{(x, y, z) \in \mathbb{R}^3 \subset \mathbb{C}^3; \ x^2 + y^2 \neq 1\},
\]

is the rational function of three variables.

**Lemma 4.3.1.** Let \( P, Q \) be polynomials of the form (1) (of one variable and with real coefficients) and let \( \deg P < \deg Q \). Let \( Q \) have a real root \( a \) of multiplicity \( k \) \((k \geq 1)\) and such that

\[
Q(x) = (x - a)^k Q_1(x), \ x \in \mathbb{C}, \ \text{where} \ Q_1(a) \neq 0.
\]

Then there is a real number \( A \) and a polynomial \( P_1 \) such that

\[
\frac{P(x)}{Q(x)} = \frac{A}{(x - a)^k} + \frac{P_1(x)}{(x - a)^{k-1}Q_1(x)}, \ x \in \{y \in \mathbb{C} : Q(y) \neq 0 \} =: M,
\]

and \( \deg P_1 < \deg Q - 1 \).

**Proof.** For \( A \in \mathbb{R} \) and \( x \in M \) we have

\[
\frac{P(x)}{Q(x)} = \frac{A}{(x - a)^k} + \frac{P(x) - AQ_1(x)}{(x - a)^{k-1}Q_1(x)}.
\]

Since \( Q_1(a) \neq 0 \), we have that there exists \( A \in \mathbb{R} \) such that \( P(a) - AQ_1(a) = 0 \). If the polynomial \( x \mapsto P(x) - AQ_1(x), \ x \in \mathbb{C} \) is not zero then \( P(x) = (x - a)P_1(x), \ x \in \mathbb{C} \) and \( \deg P_1 = \deg(P - AQ_1) - 1 \leq \max[\deg P, \deg Q_1] - 1 < \deg Q - 1. \)

**Lemma 4.3.2.** Let \( P, Q \) be polynomials of the form (1) (of one variable and with real coefficients) and let \( \deg P < \deg Q \). Let \( Q \) have a root \( z \in \mathbb{C} \) of multiplicity \( k \) \((k \geq 1)\) and such that

\[
Q : x \mapsto (x - z)^k (x - \bar{z})^k Q_1(x), \ x \in \mathbb{C}, \ \text{where} \ Q_1(z) \neq 0 \neq Q_1(\bar{z}).
\]
Then there exists real numbers $B, C$ and a polynomial $P_1$ such that
\[
\frac{P(x)}{Q(x)} = \frac{Bx + C}{(x - z)^k(x - \bar{z})^k} + \frac{P_1(x)}{(x - z)^{(k-1)}(x - \bar{z})^{(k-1)}Q_1(x)}, \quad x \in \{y \in \mathbb{C} : Q(y) \neq 0\} =: M,
\]
and
\[
\text{deg } P_1 < \text{deg } Q - 2.
\]

\textbf{Proof.} By using Theorem 4.3.5, the proof is very similar to that of Lemma 4.3.1. \hfill \Box

\textbf{Definition 4.3.5.} Let $A, B, C$ be real numbers and let $a \in \mathbb{R}$ and $z \in \mathbb{C}$ and $k \in \mathbb{N}$. The rational functions
\[
x \mapsto \frac{A}{(x - a)^k}, \quad x \in \mathbb{C} \setminus \{a\} \quad \text{and} \quad x \mapsto \frac{C x + C}{(x - z)^k(x - \bar{z})^k}, \quad x \in \mathbb{C} \setminus \{z, \bar{z}\}
\]
are called the \textit{partial fractions}.

\textbf{Theorem 4.3.6.} Let $P, Q$ be polynomials of the form (1) (of one variable and with real coefficients) and let $\text{deg } P < \text{deg } Q$. Let $Q$ has the factorization (2). Then there exist real numbers $A_1, \ldots, A_{k_1}; \ldots; A_1, \ldots, A_{k_2}; P_1^1, Q_1^1; \ldots; P_1^l, Q_1^l; \ldots; P_i^j, Q_i^j$ (their number is equal to the degree of the polynomial $Q$) such that for any $x \in \mathbb{C} \setminus \{y \in \mathbb{C} : Q(y) = 0\}$ we have
\[
\frac{P(x)}{Q(x)} = \left[ \frac{A_{k_1}^1}{(x - x_1)} + \cdots + \frac{A_{k_1}^1}{x - x_1} \right] + \cdots + \left[ \frac{A_{k_2}^1}{(x - x_1)} + \cdots + \frac{A_{k_2}^1}{x - x_1} \right] +
\left[ \frac{P_1^1 x + Q_1^1}{(x^2 + p_1 x + q_1)^{l_1}} + \cdots + \frac{P_1^l x + Q_1^l}{x^2 + p_1 x + q_1} \right] + \cdots + \left[ \frac{P_i^j x + Q_i^j}{(x^2 + p_i x + q_i)^{l_i}} + \cdots + \frac{P_i^j x + Q_i^j}{x^2 + p_i x + q_i} \right].
\]

\textbf{Proof.} The proof can be done by induction with respect to the degree of the polynomial $Q$, however it is technically quite complicated and we will omit it. We refer the reader to the literature \cite{6}, chapter IV, section 2, where you can find a detailed proof. \hfill \Box

We will demonstrate the usage of the expansion from Theorem 4.3.6 in the following example in which we will see how to find the coefficients of the expansion into the partial fractions.

\textbf{Example 4.3.2.} Expand into the partial fractions the following rational function:
\[
f : x \mapsto \frac{2x^6}{x^6 + x^4 - x^2 - 1}, \quad x \in \mathbb{C} \setminus \{1, -1, i, -i\}.
\]
Solution: If we denote the nominator by $P$ and the denominator by $Q$ then we know, from Example 4.3.1, that

$$Q(x) = (x - 1)(x + 1)(x^2 + 1)^2,$$

and

$$P(x) = 2Q(x) - 2x^4 + 2x^2 + 2.$$

So, by division of polynomials, we get

$$\frac{P(x)}{Q(x)} = 2 + \frac{-2x^4 + 2x^2 + 2}{Q(x)},$$

and the expansion of the right-hand side of the previous equation into the partial fractions has the form

$$\frac{-2x^4 + 2x^2 + 2}{Q(x)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{Cx + D}{(x^2 + 1)^2} + \frac{Ex + F}{x^2 + 1}.$$

This exactly means that

$$-2x^4 + 2x^2 + 2 = A(x^5 + x^4 + 2x^3 + x + 1) + B(x^5 - x^4 + 2x^3 - 2x^2 + x - 1) + (Cx + D)(x^2 - 1) + (Ex + F)(x^4 - 1).$$

With respect to Theorem 4.3.4 the coefficients $A, B, \ldots, F$ must obey the system of (linear) equations that arises from comparison of terms by equal powers of the independent variable $x$. The resulting system reads:

\[
\begin{align*}
0 &= A + B + E \\
-2 &= A - B + F \\
0 &= 2A + 2B + C \\
2 &= 2A - 2B + D \\
0 &= A + B - C - E \\
2 &= A - B - D - F.
\end{align*}
\]

The (only) solution to this system is given by

$$A = \frac{1}{4}, \; B = -\frac{1}{4}, \; C = 0, \; D = 1, \; E = 0, \; F = -\frac{5}{2}.$$ 

Thus, for any $x \in \mathbb{C} \setminus \{1, -1, i, -i\}$ we have

$$\frac{2x^6}{x^6 + x^4 - x^2 - 1} = 2 + \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} + \frac{1}{(x^2 + 1)^2} - \frac{5}{2(x^2 + 1)}.$$
4.4 Elementary functions

Problems

1. Find the factorization of the polynomials:
   a) \( Q : x \mapsto x^9 + 2x^6 + x^3, \ x \in \mathbb{C}; \)
   b) \( S : x \mapsto x^4 + 1, \ x \in \mathbb{C}. \)

2. Find the expansion of the polynomials following rations functions into the partial fractions:
   a) 
   \[
   x \mapsto \frac{x^7 + 7x - 1}{Q(x)}, \quad x \in \mathbb{C} \setminus \{ y \in \mathbb{C} : Q(y) \neq 0 \},
   \]
   b) 
   \[
   x \mapsto \frac{1}{S(x)}, \quad x \in \mathbb{C} \setminus \{ y \in \mathbb{C} : S(y) \neq 0 \},
   \]
   where \( Q \) and \( S \) are the polynomials from Problem 1.

Answers

1  a) \( Q(x) = x^3(x + 1)^2(x^2 - x + 1)^2, \ x \in \mathbb{C} \)
   b) \( S(x) = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1), \ x \in \mathbb{C} \)

2  a) 
   \[
   -\frac{1}{x^3} + \frac{7}{x^2} + \frac{1}{(x + 1)^2} + \frac{31}{9(x + 1)} - \frac{x + 7}{3(x^2 - x + 1)^2} - \frac{31x + 1}{9(x^2 - x + 1)},
   \]
   for 
   \( x \in \mathbb{C} : x \neq 0, x \neq -1, x \neq \frac{1 \pm i\sqrt{3}}{2}. \)
   b) 
   \[
   \frac{1}{2\sqrt{2}} \left( \frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} \right),
   \]
   for 
   \( x \in \mathbb{C} : x \neq \frac{1}{\sqrt{2}}(1 \pm i), x \neq \frac{1}{\sqrt{2}}(-1 \pm i). \)

4.4 Elementary functions

In this section, we will be interested in a very special and important class of the so-called basic elementary functions. We can construct, using the functions of this class and certain operations, the wide class of elementary functions. In fact, in theory as well as in practice, we can often meet also non-elementary functions represented by functional series. However, the study of non-elementary functions exceeds this book.
4.4 Elementary functions

4.4.1 Basic elementary functions

It is supposed that the reader has some skill in using the elementary functions from previous courses in mathematics. The following definition contains the list of basic elementary functions we will be interested in.

**Definition 4.4.1.** The following mappings are called the *basic elementary functions*:

- \( x \mapsto a^x, \ x \in \mathbb{R}, \ a \in \mathbb{R}^+ \setminus \{1\} \) - the exponential function
- \( x \mapsto \log_a(x), \ x \in \mathbb{R}^+, \ a \in \mathbb{R}^+ \setminus \{1\} \) - the logarithmic function
- \( x \mapsto x^\alpha, \ x \in \mathbb{R}^+, \ \alpha \in \mathbb{R} \) - the power function
- \( x \mapsto \sin(x), \ x \in \mathbb{R} \) - the sine function
- \( x \mapsto \cos(x), \ x \in \mathbb{R} \) - the cosine function
- \( x \mapsto \tan(x), \ x \in \{v \in \mathbb{R} : \cos(v) \neq 0\} \) - the tangent function
- \( x \mapsto \cot(x), \ x \in \{v \in \mathbb{R} : \sin(v) \neq 0\} \) - the cotangent function
- \( x \mapsto \arcsin(x), \ x \in [-1, 1] \) - the arcsine function
- \( x \mapsto \arccos(x), \ x \in [-1, 1] \) - the arccosine function
- \( x \mapsto \arctan(x), \ x \in \mathbb{R} \) - the arctangent function
- \( x \mapsto \arccot(x), \ x \in \mathbb{R} \) - the arccotangent function

The functions \( \sin, \cos, \tan, \cot \) are called the *trigonometric functions* and the functions \( \arcsin, \arccos, \arctan, \arccot \) are called the *inverse trigonometric functions*.

Some of the basic elementary functions can be defined very easily on the set of rational numbers, e.g. the exponential function. The goniometric functions can be defined in the interval \([0, \pi/2]\) by an elementary geometry: by ratios of the lengths of sides in rectangled triangle. The values of the functions sine and cosine can be defined in the interval \([0, 2\pi]\) by help of unit circumference (see Fig. 4.2) as coordinates of the point \( C \). The values of the functions tangent and cotangent are also defined in the unit circumference (see again Fig. 4.2), namely

\[
\tan(x) = |BD| = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = |EF| = \frac{\cos(x)}{\sin(x)}.
\]

These equalities follows from the homothety of the triangles \( \triangle OAC \) and \( \triangle OBD \), and \( \triangle OBD \) and \( \triangle FEO \), respectively. Similarly, the values of the functions secant and cosecant are defined as lengths of the sides \( |OD| = 1/\cos(x) \), and \( |OF| = 1/\sin(x) \), respectively.

43
Our task will be more general - we need to define all the basic elementary functions exactly and for all possible values of their arguments. And this cannot be done by simple extension of a known function. For example, it is not evident how to define a real power of a real number despite the rational powers of a number are defined easily. The geometrical definition of the functions sine and cosine is not self-evident, since it assumes the existence of a bijective mapping of the interval \([0, 2\pi]\) onto the unit circumference.

### 4.4.2 Powers with a rational exponent

We will start with a list of properties of the powers of the positive real numbers with a rational exponent. We will need these properties in the sequel, when we will study properties of the exponential function and the power function.

Let us consider \(a \in \mathbb{R}^+\). Then the following properties hold:

1. \(a^0 = 1\)

2. For all \(n \in \mathbb{N}\) we have \(a^n = a^{n-1} \cdot a\)

3. For all \(n \in \mathbb{N}\) \((-n \in \mathbb{Z})\) we have \(a^{-n} = \frac{1}{a^n}\)

4. For all \(n \in \mathbb{N}\) we have granted the existence of the \(n\)-th root of \(a\): \(a^{1/n} = \sqrt[n]{a}\) (by Theorem 3.1.6, see also Definition 3.1.2)

5. Since for all \(m \in \mathbb{Z} \): \(a^m > 0\) we have for all \(n \in \mathbb{N}\): \(a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}\)
Bellow we continue with the properties obeyed by the powers with rational exponents (these properties follows from the well-known rules of computation with real numbers):

6. $a^r > 0$ for all $r \in \mathbb{Q}$

7. If $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ then $(ab)^r = a^rb^r$, and $(\frac{a}{b})^r = \frac{a^r}{b^r}$, for all $r \in \mathbb{Q}$

8. If $(r_1, r_2) \in \mathbb{Q}^2$ then $a^{r_1} a^{r_2} = a^{r_1+r_2}$, and $\frac{a^{r_1}}{a^{r_2}} = a^{r_1-r_2}$, and $(a^r)^2 = a^{r_1r_2}$

9. If $0 < a < b$ and $r \in \mathbb{Q}$ then $a^r < b^r$

10. If $a > 1$ and $(r_1, r_2) \in \mathbb{Q}$ are such that $r_1 < r_2$ then $a^{r_1} < a^{r_2}$. For $0 < a < 1$ one obtains $a^{r_1} > a^{r_2}$.

**Example 4.4.1.** Let $1 < a \in \mathbb{R}$. Show that

$$\forall \epsilon > 0 \exists n_0 \in \mathbb{N} : (n \geq n_0 \Rightarrow a^{1/n} - 1 < \epsilon).$$

**Solution:** Let us introduce the notation

$$\delta_n := a^{1/n} - 1.$$  

Property 10 listed above tells us that $a^{1/n} > 1$, thus $\delta_n > 0$ is positive. The binomial theorem implies that

$$a = (a^{1/n})^n = (1 + \delta_n)^n = 1 + n\delta_n + \Delta_n > 1 + n\delta_n,$$

since the remainder denoted by $\Delta_n$ is obviously positive. So, we have:

$$\delta_n < \frac{a-1}{n}.$$  

The Archimedean property (Theorem 3.1.4) ensures the existence of a natural number $n_0$ such that

$$\frac{a-1}{\epsilon} < n_0, \quad \text{or} \quad \frac{a-1}{n_0} < \epsilon.$$  

For such an $n_0$ and for all $n \geq n_0$ one obviously obtains

$$\frac{a-1}{n} \leq \frac{a-1}{n_0} < \epsilon,$$

thus, for all $n \geq n_0$ we have $0 < a^{1/n} - 1 < \epsilon$, as required.
4.4.3 Powers with a real exponent

The way we will define in the real power of a real number is motivated by Example 3.1.3.

**Definition 4.4.2.** Let $x \in \mathbb{R}$.
1. For any $a \in [1, +\infty)$ we can define

$$E := \{p \in \mathbb{Q}; \; p < x\} \text{ and } F := \{p \in \mathbb{Q}; \; x < p\}.$$ 

Then the real number $y$ that obeys:

$$\forall r \in E \forall s \in F : \; a^r \leq y \leq a^s \quad (1)$$

is called the **power of the number $a$ with real exponent $x$**. We denote the number $y$ by $a^x$.

2. If $a \in (0, 1)$, then we define:

$$a^x := \left(\frac{1}{a}\right)^{-x}, \quad (1/a > 1).$$

Having formulated this definition of the real power of a real number, it is not obvious to be sure that there is just one number $y$ with given properties. We have to prove the correctness of Definition 4.4.2.

**THEOREM 4.4.1.** Definition 4.4.2 is correct, i.e. for all $x \in \mathbb{R}$ and for all $a > 1$ there exists just one real number $y$ with property (1).

**Proof.** 1. First, we will prove the existence of the number $y$. Let us introduce the notation:

$$A := \{a^r \in \mathbb{R}; \; r \in E\}.$$ 

For all $r \in E$ and for any fixed $s \in F$ we have: $r < x < s$, $(r \in \mathbb{Q}, s \in \mathbb{Q})$, therefore (see property 10 of the powers with rational exponent) we get that $a^r < a^s$ for all $r \in E$. Thus, the set $A \subset \mathbb{R}$ is bounded from above and thus (see Theorem 3.1.2) there exists its supremum:

$$\sup A =: y \in \mathbb{R}.$$ 

The first property of the supremum implies that $\forall r \in E : \; a^r \leq y$. Since $y$ is the least upper bound of the set $A$, we have that $y \leq a^s$ for any $s \in F$. We can conclude that the number $y$ obeys (1) and this proves its existence.

2. The uniqueness will be proven indirectly. Let us suppose there are two distinct real numbers $y_1, y_2$ obeying (1), this means the following holds true:

$$\forall r \in E \forall s \in F : \; a^r \leq y_1 \leq a^s, \quad a^r \leq y_2 \leq a^s.$$ 

As a consequence we have:

$$-(a^s - a^r) \leq y_1 - y_2 \leq a^s - a^r.$$
For a fixed $s_0 \in F$ and for all $r \in E$, $s \in F$ we have

$$|y_1 - y_2| \leq a^s - a^r = a^r(a^{s-r} - 1) < a^{s_0}(a^{s-r} - 1).$$

By Example 3.1.3 (see also Example 3.1.4) we have

$$x = \sup E = \inf E.$$

Theorem 3.1.3 implies

$$\forall n \in \mathbb{N} \ \exists r_1 \in E \ \exists s_1 \in F : \ x - \frac{1}{2n} < r_1 \ & \ x + \frac{1}{2n} > s_1,$$

i.e.

$$s_1 - r_1 < \frac{1}{n}.$$ 

Thus,

$$\forall n \in \mathbb{N} : \ |y_1 - y_2| < a^{s_0}(a^{r_1} - 1) < a^{s_0}(a^{1/n} - 1).$$

Now, let $\epsilon > 0$. Then Example 4.4.1 implies that

$$\exists n_0 \in \mathbb{N} : \ a^{1/n_0} - 1 < \epsilon a^{-s_0}.$$ 

So, we have obtained the following statement:

$$\forall \epsilon > 0 \ \exists n_0 \in \mathbb{N} : \ |y_1 - y_2| < a^{s_0}(a^{1/n_0} - 1) < \epsilon,$$

or simply

$$\forall \epsilon > 0 : \ |y_1 - y_2| < \epsilon.$$ 

However, this contradicts our assumption: $y_1 \neq y_2$. Therefore, $y_1 = y_2$. \hfill \Box

The correctness of the second part of Definition 4.4.2, where the case $a \in (0, 1)$ is considered, follows from the correctness of the first part of this Definition.

The powers with a real exponent obeys the following properties (analogical with the properties of the powers with a rational exponent):

**THEOREM 4.4.2.** Let $(x_1, x_2) \in \mathbb{R}^2$ and let $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then

1. $a^{x_1} a^{x_2} = a^{x_1 + x_2}$
2. $\frac{a^{x_1}}{a^{x_2}} = a^{x_1 - x_2}$
3. $(ab)^{x_1} = a^{x_1} b^{x_1}$
4. $\left(\frac{a}{b}\right)^{x_1} = \frac{a^{x_1}}{b^{x_1}}$
5. $(a^{x_1})^{x_2} = a^{x_1 x_2}$
4.4 Elementary functions

Proof. We will prove the first of the listed properties. The proofs of the properties 2–5. can be done in a similar way.

Let \( a \in [1, +\infty) \). (For \( a \in (0, 1) \) is the statement evident from the second part of Definition 4.4.2). If \( x_1, x_2 \in \mathbb{R} \) and

\[
E := \{ (r_1, s_1) \in \mathbb{Q}^2; \ r_1 < x_1 < s_1 \}, \quad F := \{ (r_2, s_2) \in \mathbb{Q}^2; \ r_2 < x_2 < s_2 \},
\]

then, for any \((r_1, s_1) \in E\) and for any \((r_2, s_2) \in F\), we have:

\[
r_1 + r_2 < x_1 + x_2 < s_1 + s_2.
\]

Thus, one obtains from Definition 4.4.2 that

\[
\exp x_1 \leq \exp r_1 \leq \exp s_1, \quad \exp x_2 \leq \exp r_2 \leq \exp s_2, \quad \exp x_1 + x_2 \leq \exp r_1 + r_2 \leq \exp s_1 + s_2
\]

holds true for all \( (r_1, s_1) \in E \), \( (r_2, s_2) \in F \). After some algebra one can deduce from the previous inequalities that

\[
\exp x_1 + x_2 \leq \exp x_1 \exp x_2 \leq \exp x_1 + x_2 \quad \forall (r_1, s_1) \in E, \ (r_2, s_2) \in F.
\]

Finally, we can use the same as in the proof of Theorem 4.4.1, namely:

\[
\forall \epsilon > 0 : \ |\exp x_1 + x_2 - \exp x_1 \exp x_2| < \epsilon.
\]

Thus, the case \( \exp x_1 + x_2 \neq \exp x_1 \exp x_2 \) is ruled out. \( \square \)

4.4.4 Exponential function

We can formulate the following definition.

Definition 4.4.3. An exponential function \( \exp_a \) is defined as the real-valued function

\[
\exp_a: \ x \mapsto a^x, \quad x \in \mathbb{R},
\]

where \( a \in \mathbb{R}^+ \setminus \{1\} \) and the values \( a^x \) are specified in Definition 4.4.2.

Theorem 4.4.3. The exponential function \( \exp_a \) has the following properties:

1. It is increasing on \( \mathbb{R} \) for \( a \in (1, +\infty) \)

2. It is decreasing on \( \mathbb{R} \) for \( a \in (0, 1) \)

3. For all \( x \in \mathbb{R} \) and for all \( a \in \mathbb{R}^+ \setminus \{1\} \) the value \( a^x \) is positive

4. It is, for all \( a \in \mathbb{R}^+ \setminus \{1\} \), a bijective mapping from \( \mathbb{R} \) onto \( \mathbb{R}^+ \)
4.4 Elementary functions

5. For all \( a \in \mathbb{R}^+ \setminus \{1\} \) we have:

\[
\inf_{x \in \mathbb{R}} a^x = 0, \quad \sup_{x \in \mathbb{R}} a^x = +\infty \quad \text{in the extended real line.}
\]

Proof. 1. Let \( x_1, x_2 \in \mathbb{R} \) be two distinct numbers. Let us suppose \( x_1 < x_2 \). Then we can use Theorem 3.1.5 (the rational numbers are dense in the real numbers) to state that the exist two numbers \( c, d \) in \( \mathbb{Q} \) such that \( x_1 < c < d < x_2 \). Definition of the exponential function and the increase of the exponential function on \( \mathbb{Q} \) imply that \( a^{x_1} \leq a^c < a^d \leq a^{x_2} \), or

\[
(a^{x_1} - a^{x_2})(x_2 - x_1) > (a^d - a^c)(d - c) > 0.
\]

Thus, the exponential function is, in fact, increasing for \( a > 1 \).

2. Analogically, for \( 0 < a < 1 \) we have (using the fact that the exponential function \( \exp_a \) with \( a \in (0,1) \) is decreasing on \( \mathbb{Q} \)):

\[
x_1 < c < d < x_2, (c,d) \in \mathbb{Q}^2 \Rightarrow a^{x_1} = \left(\frac{1}{a}\right)^{−x_1} \geq \left(\frac{1}{a}\right)^{−c} \geq \left(\frac{1}{a}\right)^{−d} \geq \left(\frac{1}{a}\right)^{−x_2} = a^{x_2}.
\]

3. Let \( x \in \mathbb{R} \) and \( c \in \mathbb{Q} \) be such that \( c < x \). Then \( 0 < a^c < a^x \) for all \( a > 1 \). If \( 0 < a < 1 \) then there exists \( d \) in \( \mathbb{Q} \) such that \( x < d \) and \( a^x > a^d > 0 \).

4. Let us consider the case when \( a > 1 \). Following the properties 1. and 3. we see that the exponential function \( f : x \mapsto a^x, x \in \mathbb{R} \) is an injective mapping from \( \mathbb{R} \) into \( \mathbb{R}^+ \). To show that \( \exp_a \) is also a surjective mapping \( f : \mathbb{R} \mapsto \mathbb{R}^+ \) it suffices to show that

\[
\forall y \in \mathbb{R}^+ \exists x \in \mathbb{R} : a^x = y.
\]

Let us suppose the opposite is true, i.e.

\[
\exists y_0 \in \mathbb{R}^+ \forall x \in \mathbb{R} : a^x \neq y_0 \quad (a^x < y_0 \text{ or } a^x > y_0).
\]

It follows from the binomial theorem \((n \in \mathbb{N})\) that

\[
a^n = [1 + (a - 1)]^n = 1 + n(a - 1) + \Delta > n(a - 1), \quad \Delta > 0.
\]

So, there exists \( n \) in \( \mathbb{N} \) such that \( a^n > y_0 \). (It is sufficient to choose \( n \) as: \( n(a - 1) > y_0 \), i.e. \( n > y_0/(a - 1) \)). To have \( a^{-m} \leq y_0 \) it suffices to choose \( m \) as: \( 1/(m(a - 1)) \leq y_0 \) or \( \mathbb{N} \ni m \geq 1/(y_0(a - 1)) \). If it would hold true: \( a^x < y_0 \) for any real \( x \) then also \( a^x < a^n \) would be true. However, the exponential function is increasing and therefore we must have \( x < n \) for all \( x \in \mathbb{R} \) - this, however, is in contradiction with the fact that the set of all natural numbers \( \mathbb{N} \) is unbounded.

Similarly, for \( y_0 < a^x \) one must have \( a^{-m} < a^x \) and therefore for any \( x \in \mathbb{R} \) this would require \( -m < x \) that is again impossible because of the fact that the set of all real numbers \( \mathbb{R} \) is also unbounded.

49
Finally, the mapping $f$ is bijective.
The fact that $f$ is bijective in the case $a \in (0, 1)$ follows from Theorem 2.2.3 (on composite mappings). In fact,
\[ a^x = \left( \frac{1}{a} \right)^{-x}, \quad x \in \mathbb{R}, \]
can be understood as a value of the composite mapping $g \circ h : \mathbb{R} \to \mathbb{R}^+$, where $g : v \mapsto (1/a)^v$, $v \in \mathbb{R}$ and $1/a > 1$ is bijective mapping from $\mathbb{R}$ onto $\mathbb{R}^+$ and $h : x \mapsto -x$, $x \in \mathbb{R}$ is bijective mapping from $\mathbb{R}$ onto $\mathbb{R}$.

5. It holds true:
\[ \inf_{x \in \mathbb{R}} a^x := \inf(0, +\infty) = 0 \quad (\text{in } \mathbb{R}), \]
and
\[ \sup_{x \in \mathbb{R}} a^x := \sup(0, +\infty) = 0 \quad (\text{in } \overline{\mathbb{R}}). \]

The property 5 from the previous Theorem tells us that the exponential function is bounded from below by zero and it is unbounded from above. The graph of the exponential function is shown in Figure 4.3.

\[ y = a^x, \; a > 1 \]
\[ y = a^x, \; 0 < a < 1 \]

\[ \begin{align*}
\inf_{x \in \mathbb{R}} a^x := \inf(0, +\infty) &= 0 \quad (\text{in } \mathbb{R}), \\
\sup_{x \in \mathbb{R}} a^x := \sup(0, +\infty) &= 0 \quad (\text{in } \overline{\mathbb{R}}).
\end{align*} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{exp_graph.png}
\caption{Figure 4.3:}
\end{figure}

### 4.4.5 Logarithmic function

The exponential function is the bijective map from $\mathbb{R}$ onto $\mathbb{R}^+$. Therefore its inverse mapping exists.

**Definition 4.4.4.** The inverse function with respect to the exponential function $(\exp_a : x \mapsto a^x, \; x \in \mathbb{R}, \; a \in \mathbb{R}^+ \setminus \{1\})$ is called the **logarithmic function**. It is denoted by $\log_a$.

As a consequence of Definition 4.4.4 and Theorem 4.4.3 we have the following Theorem

**THEOREM 4.4.4.** The logarithmic function $\log_a : \mathbb{R}^+ \to \mathbb{R}$ has the following properties:
1. For any $a \in \mathbb{R}^+ \setminus \{1\}$ the logarithmic function is the bijective mapping.

2. For any $a > 1$ the logarithmic function is increasing on $\mathbb{R}$.

3. For any $0 < a < 1$ the logarithmic function is decreasing on $\mathbb{R}$.

4. For any $a \in \mathbb{R}^+ \setminus \{1\}$ we have:

\[
\inf_{x > 0} \log_a(x) = -\infty \quad (\text{in } \bar{\mathbb{R}}), \quad \sup_{x > 0} \log_a(x) = +\infty \quad (\text{in } \bar{\mathbb{R}}).
\]

Proof. The property 1 follows the part 4 of Theorem 4.4.3. The properties 2 and 3 follow from Theorem 4.2.2. The property 4 is a direct consequence of Definition 2.1.4.

The value of the logarithmic function $\log_a$ at the point $x \in \mathbb{R}^+$ is called the logarithm to the base $a$ of the number $x$. If we denote this logarithm by $y$ then it follows from Definition 4.4.4 that $a^y = x$, or $a^{\log_a(x)} = x$.

One uses frequently the following notation: $\log(x) := \log_{10}(x)$, for $x > 0$.

The next theorem summarizes the rules of computation with logarithms.

**THEOREM 4.4.5.** Let $a \in \mathbb{R}^+ \setminus \{1\}$, $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $z \in \mathbb{R}$, then:

1. $\log_a(xy) = \log_a(x) + \log_a(y)$.

2. $\log_a(x/y) = \log_a(x) - \log_a(y)$.

3. $\log_a(x^z) = z \log_a(x)$.

4. If furthermore $b \in \mathbb{R}^+ \setminus \{1\}$, then

\[
\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.
\]

Proof. The proof of the theorem is a simple consequence of definition of the logarithm and of Theorem 4.4.2.

The graph of the logarithmic function is shown in Figure 4.4.
4.4 Elementary functions

\[ y = \log_a(x), \ a > 1 \]

\[ y = \log_a(x), \ 0 < a < 1 \]

Figure 4.4: