Príklad:
Pomocou d'Alambertovej formuly nájdite funkciu $u = u(x,t)$ splňujúcu

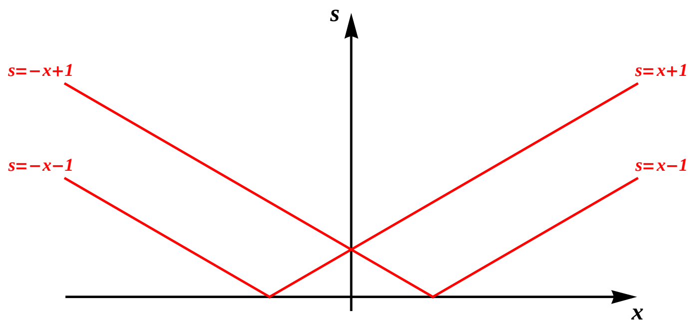
$$\begin{array}{ll} u_{tt} - u_{xx} = f(x) & x \in \mathbb{R},\ 0 < t < \infty \\ u(x,0) = 0 & \forall x \in \mathbb{R} \\ u_t(x,0) = 0 & \forall x \in \mathbb{R} \end{array} \qquad \text{kde} \quad f(x) = \begin{cases} 1 & : \quad |x| \leq 1 \\ 0 & : \quad |x| > 1\,. \end{cases}$$

Riešenie:
Z d'Alambertovej formuly vyplýva, že riešenie môžeme hľadať v tvare

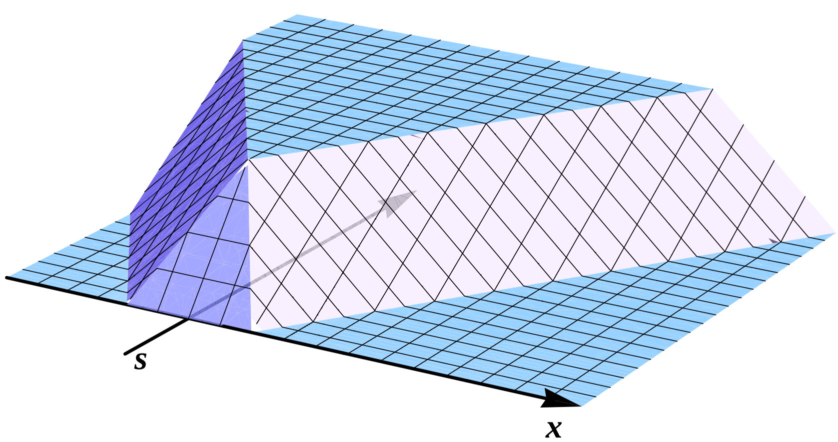
$$u(x,t) = \int_0^t \frac{F(x+s) - F(x-s)}{2} \mathrm{d}s \qquad \text{kde} \qquad F(x) = \begin{cases} 1 & : \quad x > 1 \\ x & : \quad |x| \leq 1 \\ -1 & : \quad x < -1 \end{cases}.$$

Najprv uvažujme funkciu $\qquad g(x,s) := \dfrac{F(x+s) - F(x-s)}{2}.$

Aby sme mohli nakresliť graf funkcie g, jej definičný obor potrebujeme rozdeliť na niekoľko častí (pomocou priamok, ktoré sú na obrázku nižšie vyznačené červenu farbou):

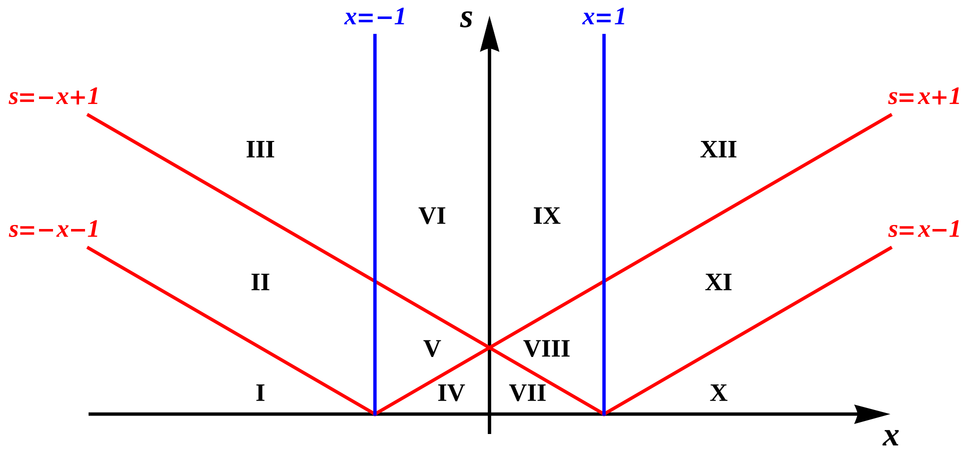


Graf funkcie g je znázornený nižšie, pričom je dobré si uvedomiť, že funkcia g je párna vzhľadom na premennú x:

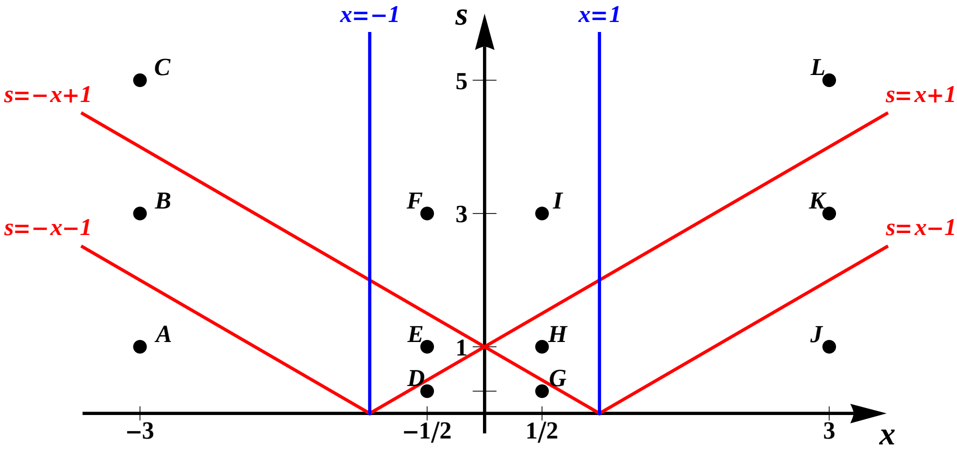


Ale aby sme mohli počítať integrál z tejto funkcie (t.j. $u(x,t) = \int_0^t g(x,s)\mathrm{d}s$), definičný obor funkcie g rozdeľme nasledovne:

$$
\begin{array}{rllll}
\text{I.} & 0 < s < -x-1 & (\Rightarrow & x < -1) & \Longrightarrow \quad g_1(x,s) = 0 \\
\text{II.} & -x-1 < s < -x+1 & \wedge & x \leq -1 & \Longrightarrow \quad g_2(x,s) = \dfrac{x+s+1}{2} \\
\text{III.} & -x+1 < s & \wedge & x \leq -1 & \Longrightarrow \quad g_3(x,s) = 1 \\
\text{IV.} & 0 < s < x+1 & \wedge & -1 < x \leq 0 & \Longrightarrow \quad g_4(x,s) = s \\
\text{V.} & x+1 < s < -x+1 & \wedge & -1 \leq x \leq 0 & \Longrightarrow \quad g_5(x,s) = \dfrac{x+s+1}{2} \\
\text{VI.} & -x+1 < s & \wedge & -1 \leq x \leq 0 & \Longrightarrow \quad g_6(x,s) = 1 \\
\text{VII.} & 0 < s < -x+1 & \wedge & 0 \leq x < 1 & \Longrightarrow \quad g_7(x,s) = s \\
\text{VIII.} & -x+1 < s < x+1 & \wedge & 0 \leq x \leq 1 & \Longrightarrow \quad g_8(x,s) = \dfrac{1-x+s}{2} \\
\text{IX.} & x+1 < s & \wedge & 0 \leq x \leq 1 & \Longrightarrow \quad g_9(x,s) = 1 \\
\text{X.} & 0 < s < x-1 & (\Rightarrow & 1 < x) & \Longrightarrow \quad g_{10}(x,s) = 0 \\
\text{XI.} & x-1 < s < x+1 & \wedge & 1 \leq x & \Longrightarrow \quad g_{11}(x,s) = \dfrac{1-x+s}{2} \\
\text{XII.} & x+1 < s & \wedge & 1 \leq x & \Longrightarrow \quad g_{12}(x,s) = 1
\end{array}
$$



Výpočet integrálu funkcie g od $s=0$ po $s=t$ rozdelíme na základe toho, v ktorej oblasti sa nachádza bod (x,t), t.j. napr. $A \in I, B \in II, \dots, L \in XII$:



I. $$u(x,t)=\int_0^t g(x,s)\mathrm{d}s=\int_0^t g_1(x,s)\mathrm{d}s=0$$

II. $$u(x,t)=\int_0^t g(x,s)\mathrm{d}s=\int_0^{-x-1} g_1(x,s)\mathrm{d}s+\int_{-x-1}^{t} g_2(x,s)\mathrm{d}s=\frac{t}{2}+\frac{xt}{2}+\frac{t^2}{4}+\frac{(x+1)^2}{4}$$

III. $$u(x,t)=\int_0^t g(x,s)\mathrm{d}s=\int_0^{-x-1} g_1(x,s)\mathrm{d}s+\int_{-x-1}^{-x+1} g_2(x,s)\mathrm{d}s+\int_{-x+1}^{t} g_3(x,s)\mathrm{d}s=t+x$$

IV. $$u(x,t)=\int_0^t g(x,s)\mathrm{d}s=\int_0^t g_4(x,s)\mathrm{d}s=\frac{t^2}{2}$$

V. $$u(x,t)=\int_0^t g(x,s)\mathrm{d}s=\int_0^{x+1} g_4(x,s)\mathrm{d}s+\int_{x+1}^{t} g_5(x,s)\mathrm{d}s=\frac{t}{2}+\frac{xt}{2}+\frac{t^2}{4}-\frac{(x+1)^2}{4}$$

VI. $$u(x,t)=\int_0^t g(x,s)\mathrm{d}s=\int_0^{x+1} g_4(x,s)\mathrm{d}s+\int_{x+1}^{-x+1} g_5(x,s)\mathrm{d}s+\int_{-x+1}^{t} g_6(x,s)\mathrm{d}s=t-\frac{x^2+1}{2}$$

VII. $$u(x,t)=\int_0^t g(x,s)\mathrm{d}s=\int_0^t g_7(x,s)\mathrm{d}s=\frac{t^2}{2}$$

VIII. $$u(x,t)=\int_0^t g(x,s)\mathrm{d}s=\int_0^{-x+1} g_7(x,s)\mathrm{d}s+\int_{-x+1}^{t} g_8(x,s)\mathrm{d}s=\frac{t}{2}-\frac{xt}{2}+\frac{t^2}{4}-\frac{(-x+1)^2}{4}$$

$$\text{IX.} \qquad u(x,t) = \int_0^t g(x,s)\mathrm{d}s = \int_0^{-x+1} g_7(x,s)\mathrm{d}s + \int_{-x+1}^{x+1} g_8(x,s)\mathrm{d}s + \int_{x+1}^{t} g_9(x,s)\mathrm{d}s = t - \frac{x^2+1}{2}$$

$$\text{X.} \qquad u(x,t) = \int_0^t g(x,s)\mathrm{d}s = \int_0^t g_{10}(x,s)\mathrm{d}s = 0$$

$$\text{XI.} \qquad u(x,t) = \int_0^t g(x,s)\mathrm{d}s = \int_0^{x-1} g_{10}(x,s)\mathrm{d}s + \int_{x-1}^{t} g_{11}(x,s)\mathrm{d}s = \frac{t}{2} - \frac{xt}{2} + \frac{t^2}{4} + \frac{(-x+1)^2}{4}$$

$$\text{XII.} \qquad u(x,t) = \int_0^t g(x,s)\mathrm{d}s = \int_0^{x-1} g_{10}(x,s)\mathrm{d}s + \int_{x-1}^{x+1} g_{11}(x,s)\mathrm{d}s + \int_{x+1}^{t} g_{12}(x,s)\mathrm{d}s = t - x$$

Výpočet by bolo možné skrátiť, ak využijeme párnosť funkcie g vzhľadom na premennú x.