Pr 1

$$A=\begin{pmatrix}1&2&3\\2&1&4\\1&1&1\end{pmatrix}\cdot(-2)\sim\begin{pmatrix}1&2&3\\0&-3&-2\\1&1&1\end{pmatrix}\cdot(-1)\sim\begin{pmatrix}1&2&3\\0&-3&2\\0&-1&-2\end{pmatrix}\cdot(-3)\sim$$

$$\sim\begin{pmatrix}1&2&3\\0&-3&2\\0&0&8\end{pmatrix}\Rightarrow h(A)=3$$

$$\det A=\begin{vmatrix}1&2&3\\2&1&4\\1&1&1\end{vmatrix}=4\Rightarrow h(A)=3 \text{ a teda aj platí}$$

Pr 2

$$A=\begin{pmatrix}-5&2&3\\-7&4&2\\-8&2&7\end{pmatrix}\begin{matrix}\cdot(-7)\\ \cdot 5\\ \\ \end{matrix}\sim\begin{pmatrix}-5&2&3\\0&6&-11\\-8&2&7\end{pmatrix}\begin{matrix}\cdot(-8)\\ \\ \cdot 5\end{matrix}\sim\begin{pmatrix}-5&2&3\\0&6&-11\\0&-6&11\end{pmatrix}\sim$$

$$\sim\begin{pmatrix}-5&2&3\\0&6&-11\\0&0&0\end{pmatrix}\Rightarrow h(A)=2$$

$$\det A=\begin{vmatrix}-5&2&3\\-7&4&2\\-8&2&7\end{vmatrix}=0\Rightarrow h(A)<3 \text{ a teda je } h=2.$$

Pr 3

$$A=\begin{pmatrix}2&3&1\\6&9&3\end{pmatrix}\cdot(-3)\sim\begin{pmatrix}2&3&1\\0&0&0\end{pmatrix}\Rightarrow h(A)=1$$

$$B=\begin{pmatrix}1&2&-2\\2&6&-5\end{pmatrix}\cdot(-2)\sim\begin{pmatrix}1&2&3\\0&2&-1\end{pmatrix}\Rightarrow h(B)=2$$

Pr 4) $A = \begin{pmatrix} 3 & 3 \\ -1 & -2 \end{pmatrix}$; $A^{-1} = ?$

1. spôsob: $A^{-1} = \frac{1}{\det A} \begin{pmatrix} \bar{A}_{11} & \bar{A}_{21} \\ \bar{A}_{12} & \bar{A}_{22} \end{pmatrix}$; $\bar{A}_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$

$\det A = \begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} = 3 \cdot (-2) - (-1) \cdot 3 = -3$

$$\left.\begin{aligned} \bar{A}_{11} &= (-1)^{1+1} \cdot (-2) = -2 \\ \bar{A}_{12} &= (-1)^{1+2} \cdot (-1) = 1 \\ \bar{A}_{21} &= (-1)^{2+1} \cdot 3 = -3 \\ \bar{A}_{22} &= (-1)^{2+2} \cdot 3 = 3 \end{aligned}\right\} \Rightarrow \quad A^{-1} = \frac{1}{-3} \begin{pmatrix} -2 & -3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 1 \\ -\frac{1}{3} & -1 \end{pmatrix}$$

2. spôsob:

$$\left(\begin{array}{cc|cc} 3 & 3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{array}\right) \cdot 3 \; + \sim \left(\begin{array}{cc|cc} 3 & 3 & 1 & 0 \\ 0 & -3 & 1 & 3 \end{array}\right) + \sim \left(\begin{array}{cc|cc} 3 & 0 & 2 & 3 \\ 0 & -3 & 1 & 3 \end{array}\right) \begin{array}{l} :3 \\ :(-3) \end{array} \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & 1 \\ 0 & 1 & -\frac{1}{3} & -1 \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} \frac{2}{3} & 1 \\ -\frac{1}{3} & -1 \end{pmatrix}$$

Gaussova eliminační metoda:

Př 1

$$\begin{aligned} 3x+2y-z&=8\\ -x+3y+2z&=3\\ 2x-y+4z&=-4 \end{aligned} \Rightarrow \begin{pmatrix} 3 & 2 & -1\\ -1 & 3 & 2\\ 2 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 8\\ 3\\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 8\\ -1 & 3 & 2 & 3\\ 2 & -1 & 4 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 3 & 2 & 3\\ 3 & 2 & -1 & 8\\ 2 & -1 & 4 & -4 \end{array}\right) \begin{matrix} \cdot 3 \;\; \cdot 2 \\ + \\ + \end{matrix} \sim \left(\begin{array}{ccc|c} -1 & 3 & 2 & 3\\ 0 & 11 & 5 & 17\\ 0 & 5 & 8 & 2 \end{array}\right) \begin{matrix} \\ \cdot(-5) \\ \cdot 11 \;+ \end{matrix} \sim$$

$$\sim \left(\begin{array}{ccc|c} -1 & 3 & 2 & 3\\ 0 & 11 & 5 & 17\\ 0 & 0 & 63 & -63 \end{array}\right) :63 \sim \left(\begin{array}{ccc|c} -1 & 3 & 2 & 3\\ 0 & 11 & 5 & 17\\ 0 & 0 & 1 & -1 \end{array}\right) \begin{matrix} + \\ + \\ \cdot(-5) \;\; \cdot(-2) \end{matrix} \sim \left(\begin{array}{ccc|c} -1 & 3 & 0 & 5\\ 0 & 11 & 0 & 22\\ 0 & 0 & 1 & -1 \end{array}\right) :11 \sim$$

$$\sim \left(\begin{array}{ccc|c} -1 & 3 & 0 & 5\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1 \end{array}\right) \begin{matrix} + \\ \cdot(-3) \\ \\ \end{matrix} \sim \left(\begin{array}{ccc|c} -1 & 0 & 0 & -1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1 \end{array}\right) :(-1) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} -1\\ 2\\ -1 \end{pmatrix} \Rightarrow \boxed{\begin{aligned} x&=-1\\ y&=2\\ z&=-1 \end{aligned}}$$

Př 2

$$\begin{aligned} y+2z&=-6\\ 2x-y-z&=4\\ 3x+y-z&=9\\ 5x+2y\quad&=9 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 2 & -6\\ 2 & -1 & -1 & 4\\ 3 & 1 & -1 & 9\\ 5 & 2 & 0 & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4\\ 0 & 1 & 2 & -6\\ 3 & 1 & -1 & 9\\ 5 & 2 & 0 & 9 \end{array}\right) \begin{matrix} \cdot(-3) \;\; \cdot(-5) \\ + \\ \cdot 2 \;\; + \\ \cdot 2 \end{matrix} \sim$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4\\ 0 & 1 & 2 & -6\\ 0 & 5 & 1 & 6\\ 0 & 9 & 5 & -2 \end{array}\right) \begin{matrix} \\ \cdot(-5) \;\; \cdot(-9) \\ + \\ + \end{matrix} \sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4\\ 0 & 1 & 2 & -6\\ 0 & 0 & -9 & 36\\ 0 & 0 & -13 & 52 \end{array}\right) \begin{matrix} \\ \\ :(-9) \\ :(-13) \end{matrix} \sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4\\ 0 & 1 & 2 & -6\\ 0 & 0 & 1 & -4\\ 0 & 0 & 1 & -4 \end{array}\right) \begin{matrix} \\ \\ \\ + \end{matrix} \sim$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4\\ 0 & 1 & 2 & -6\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} 2 & -1 & -1\\ 0 & 1 & 2\\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 4\\ -6\\ -4 \end{pmatrix} \Rightarrow \begin{aligned} 2x-y-z&=4\\ y+2z&=-6\\ z&=-4 \end{aligned} \Rightarrow$$

$$\Rightarrow y+2(-4)=-6 \Rightarrow y=2 \Rightarrow 2x-2-(-4)=4 \Rightarrow x=1$$

$$\Rightarrow \boxed{(x, y, z)=(1, 2, -4)}$$

Pr 3

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 0 \\ 3x_1 - x_2 + x_3 + x_4 &= 1 \\ x_1 + 8x_2 - 13x_3 + 7x_4 &= 2 \end{aligned} \Rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 3 & -1 & 1 & 1 & 1 \\ 1 & 8 & -13 & 7 & 2 \end{array}\right) \begin{array}{l} \cdot(-3) \; \cdot(-1) \\ + \quad + \\ \end{array} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 0 & 5 & -8 & 4 & 1 \\ 0 & 10 & -16 & 8 & 2 \end{array}\right) \begin{array}{l} \\ \cdot(-2) \\ + \end{array} \sim \left(\begin{array}{cccc|c} 1 & -2 & +3 & -1 & 0 \\ 0 & 5 & -8 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 0 \\ 5x_2 - 8x_3 + 4x_4 &= 1 \end{aligned}$$

nekonečne veľa riešení. Budeme mať 2 parametre

napr. $x_3 = a,\ x_4 = b \Rightarrow 5x_2 - 8a + 4b = 1 \Rightarrow$

$$\Rightarrow x_2 = \frac{1+8a-4b}{5} \Rightarrow x_1 - 2\cdot\frac{1+8a-4b}{5} + 3a - b = 0 \Rightarrow$$

$$\Rightarrow x_1 = \frac{2+a-3b}{5} \Rightarrow \text{napr. ak } a = 5 \quad b = 0$$

$$\Rightarrow (x_1, x_2, x_3, x_4) = \left(\frac{7}{5}, \frac{41}{5}, 5, 0\right)$$

napr. ak $a = 0$ $b = 1$ $\Rightarrow (x_1, x_2, x_3, x_4) = \left(-\frac{1}{5}, \frac{-3}{5}, 0, 1\right)$

...

Pr 4

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 0 \\ 3x_1 - x_2 + x_3 + x_4 &= 1 \\ x_1 + 8x_2 - 13x_3 + 7x_4 &= 5 \end{aligned} \Rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 3 & -1 & 1 & 1 & 1 \\ 1 & 8 & -13 & 7 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 0 & 5 & -8 & 4 & 1 \\ 0 & 10 & -16 & 8 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 0 & 5 & -8 & 4 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right) \Rightarrow \begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 0 \\ 5x_2 - 8x_3 + 4x_4 &= 1 \\ ! \; 0 &= 3 \; ! \end{aligned}$$

$\Rightarrow \nexists$ riešenie

Frobeniova věta:

Příklady z části na Gaussovu eliminační metodu:

$\boxed{\text{Př 1}}$ úprava na trojúhelníkovou matici:

$$(A|B) = \left(\begin{array}{ccc|c} -1 & 3 & 2 & 3 \\ 0 & 11 & 5 & 17 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$h(A|B) = 3$

$h(A) = 3$

$\Rightarrow \exists$ řešení

$\boxed{\text{Př 2}}$

$$(A|B) = \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$h(A|B) = 3$

$h(A) = 3$

$\Rightarrow \exists$ řešení

$\boxed{\text{Př 3}}$

$$(A|B) = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 0 & 5 & -8 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$h(A|B) = 2$

$h(A) = 2$

$\Rightarrow \exists$ řešení

$\boxed{\text{Př 4}}$

$$(A|B) = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 0 & 5 & -8 & 4 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

$h(A|B) = 3$

$h(A) = 2$

$\neq \Rightarrow \nexists$ řešení

Cramerovo pravidlo:

$$x_1 + 2x_2 - 3x_3 = -5$$
$$3x_1 - 4x_2 + 5x_3 = 10$$
$$2x_1 + 5x_2 - 7x_3 = -9$$

$$\det A = \begin{vmatrix} 1 & 2 & -3 \\ 3 & -4 & 5 \\ 2 & 5 & -7 \end{vmatrix} = 28 + 20 - 45 - 24 - 25 + 42 = -4 \neq 0$$

$\Rightarrow$ A je regulárna matica

1. stĺpec

$$D_1 = \begin{vmatrix} -5 & 2 & -3 \\ 10 & -4 & 5 \\ -9 & 5 & -7 \end{vmatrix} = -140 - 90 - 150 + 108 + 125 + 140 = -7$$

2. stĺpec

$$D_2 = \begin{vmatrix} 1 & -5 & -3 \\ 3 & 10 & 5 \\ 2 & -9 & -7 \end{vmatrix} = -70 - 50 + 81 + 60 + 45 - 105 = -39$$

3. stĺpec

$$D_3 = \begin{vmatrix} 1 & 2 & -5 \\ 3 & -4 & 10 \\ 2 & 5 & -9 \end{vmatrix} = 36 + 40 - 75 - 40 - 50 + 54 = -35$$

$$\Rightarrow x_1 = \frac{D_1}{\det A} = \frac{7}{4} \; ; \; x_2 = \frac{D_2}{\det A} = \frac{39}{4} \; ; \; x_3 = \frac{D_3}{\det A} = \frac{35}{4}$$

Pomocou inverznej matice:

$$x_1 + x_2 + x_3 = 1$$
$$2x_1 + x_2 + 2x_3 = 1$$
$$x_1 + x_2 + 3x_3 = 2$$

$$\Rightarrow (A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ 2 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$X = A^{-1} \cdot B = \begin{pmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ 2 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + 1 - 1 \\ 2 - 1 + 0 \\ -\frac{1}{2} + 0 + 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} \Rightarrow$$

$$\Rightarrow x_1 = -\frac{1}{2} \quad x_2 = 1 \quad x_3 = \frac{1}{2}$$