doˇkaz základnej vety algebry:

nech

$$f(x) = x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

$$a_i \in \mathbb{C}, \quad 1 \le i \le n$$

$$x = r \cdot (\cos\theta + i \sin\theta) \qquad 0 \le r < +\infty, \quad 0 \le \theta < 2\pi$$

$$x^k = r^k (\cos k\theta + i \sin k\theta)$$

potom

$$f(x) = P + iQ$$

kde

$$P = r^n \cos n\theta + \dots, \qquad Q = r^n \sin n\theta + \dots$$

Výrok, že $\exists\ \hat{x}: f(\hat{x}) = 0$ je ekvivalentný s tým, že funkcia $P^2 + Q^2$ má nulový bod.

Uvážime funkciu:

$$U = \arctan\left(\frac{P}{Q}\right)$$

potom:

$$\frac{\partial U}{\partial r} = \frac{\frac{\partial P}{\partial r} Q - P \cdot \frac{\partial Q}{\partial r}}{P^2 + Q^2}$$

$$\frac{\partial U}{\partial \theta} = \frac{\frac{\partial P}{\partial \theta} Q - P \frac{\partial Q}{\partial \theta}}{P^2 + Q^2}$$

$$\frac{\partial^2 U}{\partial r \partial \theta} = \frac{H(r,\theta)}{(P^2+Q^2)^2}$$

kde H je spojitá f-cia argumentov r a θ.

~~uvažujme~~ uvážime 2 integrály:

$$I_1 = \int_0^R dr \int_0^{2\pi} d\theta \; \frac{\partial^2 U}{\partial r \partial \theta}$$

$$I_2 = \int_0^{2\pi} d\theta \int_0^R dr \; \frac{\partial^2 U}{\partial r \partial \theta}$$

kde $R > 0$. <u>Ak</u> by $P^2 + Q^2 > 0$ <u>všade</u> tak by muselo byť $I_1 = I_2 \;\; \forall R$. Ukážeme, že to tak <u>NIE JE</u> a to bude znamenať, že $P^2 + Q^2 = 0$ niekde (pre dosť veľké R). A to znamená, že platí základná veta algebry!

• výpočet I_1:

$$\int_0^{2\pi} d\theta \; \frac{\partial^2 U}{\partial r \partial \theta} = \frac{\partial U}{\partial r}\Big|_{\theta=0}^{\theta=2\pi} = 0$$

lebo $\frac{\partial U}{\partial r}$ je 2π-periodická f-cia uhla θ.

a teda $\boxed{I_1 = 0, \; \forall R}$!

• výpočet I_2:

$$\int_0^R \frac{\partial^2 U}{\partial r \partial \theta} dr = \frac{\partial U}{\partial \theta}\Big|_{r=0}^{r=R}$$

teraz:

$$\frac{\partial P}{\partial \theta} = - n r^n \sin n\theta + \dots \qquad \text{(nižšie mocniny } r\text{)}$$

$$\frac{\partial Q}{\partial \theta} = + n r^n \cos n\theta + \dots \qquad \text{(-//-)}$$

takže:

$$\frac{\partial P}{\partial \theta} \cdot Q - P \cdot \frac{\partial Q}{\partial \theta} = -n r^{2n} + \dots$$

a zase

$$P^2 + Q^2 = r^{2n} + \dots$$

takže:

$$(*) \quad \frac{\partial U}{\partial \theta} = \frac{-n r^{2n} + \dots}{r^{2n} + \dots} \xrightarrow{r \to \infty} -n$$

pri $r = 0$ je $\frac{\partial P}{\partial \theta} = \frac{\partial Q}{\partial \theta} = 0$ (o $P^2 + Q^2$ predp. > 0)

takže $\left.\frac{\partial U}{\partial \theta}\right|_{r=0} = 0$

Koeficienty v podiele $\frac{-n r^{2n} + \dots}{r^{2n} + \dots}$ (koeficienty pri r^k)

sú ohraničené funkcie (θ) $\Rightarrow$ limita $(*)$ je

rovnomerná vzhľadom na θ a preto:

$$\lim_{R \to +\infty} I_2 = \int_0^{2\pi} d\theta \, (-n) = -2\pi n$$

$\Rightarrow 0 = I_1 \neq I_2$ pre isté dosť veľké $R \Rightarrow$

$\Rightarrow$ základná veta algebry platí. /✓.