Frullaniho vzorec :

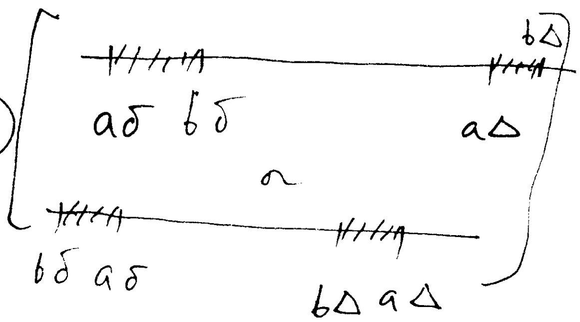
Predpoklad: $f:[0,\infty)\to\mathbb{R}$ je spojitá a čísla : $a,b>0$. Uvažujme

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = \lim_{\substack{\delta\to 0^+\\ \Delta\to+\infty}} \int_\delta^\Delta \frac{f(ax)-f(bx)}{x}\,dx$$

$$\int_\delta^\Delta \frac{f(ax)-f(bx)}{x}\,dx \overset{\forall \delta>0,\ \Delta>0}{=} \int_\delta^\Delta \frac{f(ax)}{x}\,dx - \int_\delta^\Delta \frac{f(bx)}{x}\,dx \circleddash$$

sub: $x\mapsto ax=z$ sub.: $x\mapsto bx=z$

$$\circleddash \int_{a\delta}^{a\Delta} \frac{f(z)}{z}\,dz - \int_{b\delta}^{b\Delta} \frac{f(z)}{z}\,dz \circleddash$$



$$\circleddash \int_{a\delta}^{b\delta} \frac{f(z)}{z}\,dz - \int_{a\Delta}^{b\Delta} \frac{f(z)}{z}\,dz$$

teda $\lim\limits_{\substack{\delta\to 0^+\\ \Delta\to+\infty}}(\ )$ sa rozkladá na 2 nezávislé

limity $\lim\limits_{\delta\to 0^+}\int_{a\delta}^{b\delta}(\ )$ a $\lim\limits_{\Delta\to+\infty}\int_{a\Delta}^{b\Delta}(\ )$.

Teraz: $\quad a\delta \le \xi \le b\delta$

$$\int_{a\delta}^{b\delta} \frac{f(z)}{z}\,dz = f(\xi)\int_{a\delta}^{b\delta} \frac{dz}{z} = f(\xi)\ln\left(\frac{b}{a}\right) \to$$

$$\xrightarrow[\uparrow]{\delta\to 0^+} \underline{\underline{f(0)\ln\left(\frac{b}{a}\right)}}$$

lebo f je spojitá v 0

Platia postupne (vyššie uvedené predp. stále PLATIA!):

<u>Veta 1</u>: ak $\exists \lim\limits_{z\to +\infty} f(z) \overset{\text{ozn.}}{=} f(+\infty)$

tak potom:

$$\boxed{\int_0^{+\infty} \frac{f(ax)-f(bx)}{x}\,dx = \ln\left(\frac{b}{a}\right)\left[f(0)-f(+\infty)\right]}$$

<u>d</u>.: doplníme výpočet:

$a\Delta \le \eta \le b\Delta$

$$\int_{a\Delta}^{b\Delta} \frac{f(z)}{z}\,dz = f(\eta)\int_{a\Delta}^{b\Delta} \frac{f(z)}{z}\,dz = f(\eta)\ln\left(\frac{b}{a}\right) \to$$

$$\xrightarrow[\exists \lim\limits_{z\to+\infty} f(z)]{\Delta\to+\infty} f(+\infty)\ln\left(\frac{b}{a}\right)$$

<u>Veta 2</u>: Ak pre každé $A>0$ existuje

$$\int_A^{\infty} \frac{f(z)}{z}\,dz \quad \text{potom:}$$

$$\boxed{\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = f(0)\ln\left(\frac{b}{a}\right)}$$

d.: $\exists \int^\infty \frac{f(z)}{z}\,dz$ t.j. $\int_A^B \frac{f(z)}{z}\,dz \xrightarrow[B>A]{A\to+\infty} 0$

preto:

$$\int_{a\Delta}^{b\Delta} \frac{f(z)}{z}\,dz \xrightarrow{\Delta\to+\infty} \underline{\underline{0}}$$

Veta 3: Ak pre každé $\varepsilon>0$ existuje

$$\int_0^\varepsilon \frac{f(z)}{z}\,dz$$

~~tak potom~~ a ak existuje $\lim\limits_{z\to+\infty} f(z) \overset{\text{ozn.}}{=}: f(+\infty)$

$$\boxed{\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = -f(+\infty)\ln\left(\frac{b}{a}\right)}$$

d.: viď. d. V1 ⊕ fakt, že:

$\exists \int_0^\varepsilon \frac{f(z)}{z}\,dz$ znamená, že

$$\int_{a\delta}^{b\delta} \frac{f(z)}{z}\,dz \xrightarrow{\delta\to0^+} 0.$$