(1) $I = \int_0^\infty \sin(x^2)\,dx$, $J = \int_0^\infty \cos(x^2)\,dx$

$$I = \left| x^2 = y \right| = \frac{1}{2}\int_0^\infty \frac{\sin(y)}{\sqrt{y}}\,dy = \left| \frac{1}{\sqrt{y}} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-yu^2}du \right| =$$

$$= \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\int_0^\infty dy\,\sin(y)\int_0^\infty du\; e^{-yu^2} =$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty du\int_0^\infty dy\; e^{-yu^2}\sin(y) =$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty du\,\frac{1}{1+u^4} = \frac{1}{\sqrt{\pi}}\cdot\frac{\pi}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$J = \ldots = \frac{1}{\sqrt{\pi}}\int_0^\infty du\int_0^\infty dy\; e^{-yu^2}\cos(y) =$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty du\,\frac{u^2}{1+u^4} = \frac{1}{\sqrt{\pi}}\,\frac{\pi}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

t.j. $\boxed{I = J = \frac{1}{2}\sqrt{\frac{\pi}{2}}}$

(2) $$F(\alpha,\beta) = \int_0^\infty \frac{1-\cos\alpha x}{x} e^{-\beta x}\, dx, \quad \alpha \in \mathbb{R}, \beta > 0$$

$$\frac{\partial F}{\partial \alpha} = \int_0^\infty \sin(\alpha x)\, e^{-\beta x}\, dx = \frac{\alpha}{\alpha^2+\beta^2} \Rightarrow$$

$$\Rightarrow F = \int \frac{\alpha}{\alpha^2+\beta^2}\, d\alpha = \frac{1}{2}\ln(\alpha^2+\beta^2) + \mu(\beta)$$

$$F(0,\beta) = 0 = \frac{1}{2}\ln(\beta^2) + \mu(\beta) \Rightarrow \mu(\beta) = -\frac{1}{2}\ln(\beta^2)$$

t.j.

$$\boxed{F(\alpha,\beta) = \frac{1}{2}\ln\left(1+\frac{\alpha^2}{\beta^2}\right)}$$

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(3) $$G(a,b,c) = \int_0^\infty \frac{\sin(ax)\sin(bx)}{x^2} e^{-cx}, \quad a,b \in \mathbb{R}, c > 0$$

$$G'_a = \int_0^\infty e^{-cx} \frac{\cos(ax)\sin(bx)}{x}\, dx =$$

$$= \frac{1}{2}\int_0^\infty e^{-cx} \frac{\sin[(a+b)x] - \sin[(a-b)x]}{x}\, dx =$$

$$= \frac{1}{2}\int_0^\infty e^{-cx} \frac{\sin[(a+b)x]}{x}\, dx - \frac{1}{2}\int_0^\infty e^{-cx} \frac{\sin[(a-b)x]}{x}\, dx$$

$$\stackrel{(=)}{} \quad \left| \int_0^\infty e^{-kx} \frac{\sin \omega x}{x}\, dx = \arctan\left(\frac{\omega}{k}\right) \quad \forall \omega \in \mathbb{R}, k>0 \right|$$

$$\stackrel{(=)}{} \underbrace{\frac{1}{2}\left[\arctan\left(\frac{a+b}{c}\right) - \arctan\left(\frac{a-b}{c}\right)\right]}$$

takže:

$$G = \frac{1}{2}\int da\left[\arctan\left(\frac{a+b}{c}\right) - \arctan\left(\frac{a-b}{c}\right)\right] =$$

$$= \left| \int \arctan(u)\,du = u\arctan(u) - \frac{1}{2}\ln(1+u^2) \right| =$$

$$= \frac{1}{2}\left\{ c\,\frac{a+b}{c}\arctan\left(\frac{a+b}{c}\right) - c\,\frac{a-b}{c}\arctan\left(\frac{a-b}{c}\right) + \right.$$

$$\left. \cancel{+} - \frac{c}{2}\ln\left((a+b)^2+c^2\right) + \frac{c}{2}\ln\left((a-b)^2+c^2\right)\right\} + \gamma =$$

$$= \frac{a+b}{2}\arctan\left(\frac{a+b}{c}\right) - \frac{a-b}{2}\arctan\left(\frac{a-b}{c}\right) +$$

$$+ \frac{c}{4}\ln\frac{(a-b)^2+c^2}{(a+b)^2+c^2} + \gamma$$

γ určíme z podmienky:

$$G(a=0, b, c) = \boxed{0 = \gamma} \Rightarrow$$

$$\boxed{G(a,b,c) = \frac{a+b}{2}\arctan\left(\frac{a+b}{c}\right) - \frac{a-b}{2}\arctan\left(\frac{a-b}{c}\right) + \frac{c}{4}\ln\frac{(a-b)^2+c^2}{(a+b)^2+c^2}}$$

(4) $$I(\alpha,\beta) = \int_0^\infty \frac{\sin(\alpha x)\cos(\beta x)}{x}\,dx =$$

$$= \frac{1}{2}\int_0^\infty \frac{\sin[(\alpha+\beta)x] + \sin[(\alpha-\beta)x]}{x}\,dx =$$

$$= \frac{1}{2}\int_0^\infty \frac{\sin[(\alpha+\beta)x]}{x}\,dx + \frac{1}{2}\int_0^\infty \frac{\sin[(\alpha-\beta)x]}{x}\,dx =$$

$$= \frac{\pi}{4}\left[\operatorname{sgn}(\alpha+\beta) + \operatorname{sgn}(\alpha-\beta)\right]$$

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(5) Ak $a, b > 0$:

$$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x}\,dx = \frac{1}{2}\int_0^\infty \frac{\cos[(a-b)x] - \cos[(a+b)x]}{x}\,dx$$

$$= \left| \int_0^\infty \frac{\cos\alpha x - \cos\beta x}{x}\,dx \overset{\alpha,\beta>0}{\underset{\text{Froullani}}{=}} \ln\frac{\beta}{\alpha} \right| =$$

$$= \frac{1}{2}\ln\frac{a+b}{|a-b|}$$

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⑥ $$J(\omega, \omega_1) = \int_0^\infty \frac{\sin(\omega x)}{x} \, \frac{\sin(\omega_1 x)}{x} \, dx$$

nech $\boxed{\omega > \omega_1 > 0}$

$$J(\omega, \omega_1) = \int_0^\infty \frac{\sin(\omega x)}{x} \, \frac{\sin(\omega_1 x)}{x} \, dx \overset{p.P.}{=}$$

$$= \left[-\frac{1}{x} \sin(\omega x)\sin(\omega_1 x) + \int \frac{\omega \cos(\omega x)\sin(\omega_1 x) + \omega_1 \sin(\omega x)\cos(\omega_1 x)}{x} \right]_0^\infty$$

$$= \int_0^\infty \frac{\omega \cos(\omega x)\sin(\omega_1 x) + \omega_1 \sin(\omega x)\cos(\omega_1 x)}{x} \, dx =$$

$= |$viď. príklad 4$)$

$$= \omega_1 \cdot \frac{\pi}{2} \quad \ldots \quad \boxed{J(\omega, \omega_1) \overset{\omega > \omega_1 > 0}{=} \frac{\pi}{2}\omega_1}$$

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⑦ $$H(\alpha) = \int_0^\infty \left(\frac{\sin \alpha x}{x}\right)^2 dx, \quad \alpha > 0 \quad (\text{je to } \cancel{\text{nepárne}} \text{ párne})$$

$$H'_\alpha = 2\int_0^\infty \frac{\sin(\alpha x)\cos(\alpha x)}{x} \, dx = |\text{viď. pr. } 4| =$$

$$= 2\,\frac{\pi}{4}\left[\operatorname{sgn}(2\alpha) + \operatorname{sgn}(\alpha - \alpha)\right] = \frac{\pi}{2} \Rightarrow$$

$$\Rightarrow H = \frac{\pi}{2}\alpha + \text{const} \ \wedge \ H(\alpha = 0) = 0 = \text{const.} \Rightarrow$$

$$\boxed{H(\alpha) = \frac{\pi}{2}\alpha, \quad \alpha > 0}$$

(8) $$I = \int_0^\infty \frac{\sin^5 \alpha x}{x}\, dx$$

$$\sin^5 \alpha x = \frac{1}{(2i)^5}\left(e^{i\alpha x} - e^{-i\alpha x}\right)^5 =$$

$$= \frac{1}{2^5 \cdot i}\Big[e^{5i\alpha x} - 5e^{3i\alpha x} + 10e^{i\alpha x} - 10e^{-i\alpha x} +$$

$$+ 5e^{-3i\alpha x} - e^{-5i\alpha x}\Big] =$$

$$= \frac{1}{2^4}\Big[\sin(5\alpha x) - 5\sin(3\alpha x) + 10\sin(\alpha x)\Big] \Rightarrow$$

$$I = \frac{1}{2^4}\int_0^\infty \frac{1}{x}\left\{\sin(5\alpha x) - 5\sin(3\alpha x) + 10\sin(\alpha x)\right\} dx =$$

$$= \frac{1}{2^4}\,\frac{\pi}{2}\,\mathrm{sgn}(\alpha)\left\{1 - 5 + 10\right\} = \frac{6}{2^5}\pi\,\mathrm{sgn}(\alpha) =$$

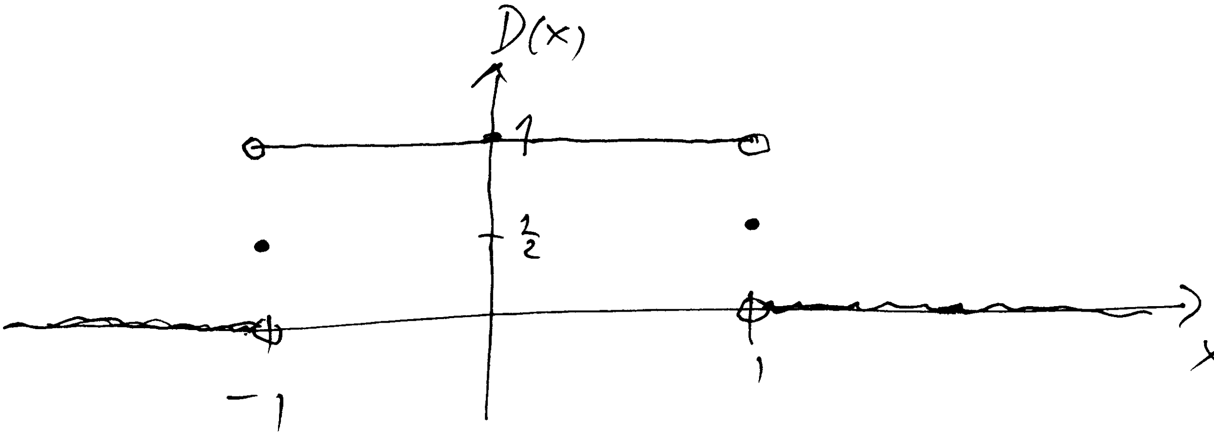
$$= \underline{\underline{\frac{3\pi}{16}\,\mathrm{sgn}(\alpha)}}$$

(9) $$D(x) = \frac{2}{\pi}\int_0^\infty \sin(\lambda)\cos(\lambda x)\,\frac{d\lambda}{\lambda} =$$

= (vidi pr. 4)

$$= \frac{2}{\pi}\cdot\frac{\pi}{4}\left\{\operatorname{sgn}(1+x) + \operatorname{sgn}(1-x)\right\} =$$

$$= \frac{1}{2}\left\{\operatorname{sgn}(1+x) + \operatorname{sgn}(1-x)\right\}$$



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(10) $$\int_0^\infty \frac{\sin^2(\alpha x) - \sin^2\beta x}{x}\,dx = \int_0^\infty \frac{\frac{1}{2} - \frac{1}{2}\cos(2\alpha x) - \frac{1}{2} + \frac{1}{2}\cos(2\beta x)}{x}\,dx =$$

$$= \frac{1}{2}\int_0^\infty \frac{\cos 2\beta x - \cos 2\alpha x}{x}\,dx \overset{\text{Froullani}}{=} \frac{1}{2}\ln\left|\frac{\alpha}{\beta}\right|$$
