Výpočet int. záv. od parametru

(1) $$F(a,b) = \int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right)^2 dx \quad , \quad a,b > 0$$

$$F_a' = 2\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \cdot \frac{e^{-ax}}{x} \cdot (-x)\, dx =$$

$$= -2\int_0^\infty \frac{e^{-2ax} - e^{-(a+b)x}}{x}\, dx \;\; (=) \;\; \left| \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx \overset{\alpha,\beta>0}{=} \ln\frac{\beta}{\alpha} \right|$$

$$(=) \; -2\ln\left(\frac{a+b}{2a}\right) = -2\ln(a+b) + 2\ln(2) + 2\ln(a)$$

$$F_b' = 2\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \; \frac{e^{-bx}}{x}\, x\, dx =$$

$$= 2\int_0^\infty \frac{e^{-(a+b)x} - e^{-2bx}}{x}\, dx = 2\ln\left(\frac{2b}{a+b}\right) =$$

$$= 2\ln 2 + 2\ln(b) - 2\ln(a+b)$$

$$F(a,b) = \int da\, F_a' = \int da \{2\ln(2) + 2\ln(a) - 2\ln(a+b)\} =$$

$$= g(b) + 2\ln(2)a + 2a\ln(a) - 2a -$$
$$-2[a\ln(a+b) - a + b\ln(a+b)] =$$

$$= g(b) + 2\ln(2)a + 2a\ln(a) - 2a - 2a\ln(a+b) + 2a$$
$$- 2b\ln(a+b)$$

tedy:

$$F'_b(a,b) = \boxed{g'(b) - \frac{2a}{a+b} - 2\ln(a+b) - \frac{2b}{a+b} = 2\ln(2) + 2\ln(b) - 2\ln(a+b)}$$

$$g'(b) - 2\ln(a+b) - \frac{2(a+b)}{a+b} = \boxed{g'(b) - \cancel{2\ln(a+b)} - 2 = 2\ln(2) + 2\ln(b) - \cancel{2\ln(a+b)}}$$

$$\Downarrow$$

$$g'(b) = 2 + 2\ln(2) + 2\ln(b)$$

$$g(b) = (2 + 2\ln(2))b + 2b\ln b - 2b + K$$

tj.

$$F(a,b) = (\cancel{2} + 2\ln(2))b + 2b\ln(b) - \cancel{2b} + 2\ln(2)a + 2a\ln(a) - \cancel{2a} - 2a\ln(a+b) + \cancel{2a} - 2b\ln(a+b) + K =$$

$$= 2\ln(2)(a+b) + 2b\ln(b) + 2a\ln(a) - 2(a+b)\ln(a+b) + K$$

$\forall a > 0$:

$$F(a,a) \overset{!}{=} \underline{0} = 2\ln(2)\cdot 2a + 2a\ln a + 2a\ln a -$$

$$- 2\cdot 2a\ln(2a) + K =$$

$$= 4\cancel{\ln(2)\cdot a} + \cancel{4a\ln(a)} - \cancel{4a\ln(2)} - \cancel{4a\ln(a)}$$

$$+ K$$

$$= \underline{K} \quad \text{tj. } K = 0$$

a tedy:

$$\boxed{F(a,b) = 2\ln(2)(a+b) + 2b\ln(b) + 2a\ln(a) - 2(a+b)\ln(a+b)}$$

② $a,b>0,\ m\in\mathbb{R}$

$$G(a,b,m)=\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\cos(mx)\,dx=$$

$$=\int_0^\infty dx\cos(mx)\int_a^b dy\, e^{-xy}=\int_a^b dy\int_0^\infty dx\, e^{-xy}\cos(mx)=$$

$$=\frac{1}{2}\int_a^b dy\int_0^\infty dx\left[e^{+x(-y+im)}+e^{x(-y-im)}\right]=$$

$$=\frac{1}{2}\int_a^b dy\left[\frac{e^{x(-y+im)}}{-y+im}-\frac{e^{x(-y-im)}}{y+im}\right]_{x=0}^{x=+\infty}=$$

$$=\frac{1}{2}\int_a^b dy\left[-\frac{1}{-y+im}+\frac{1}{y+im}\right]=$$

$$=\frac{1}{2}\int_a^b dy\,\frac{-y-im-y+im}{-y^2-m^2}=\frac{1}{2}2\int_a^b\frac{y}{y^2+m^2}\,dy=$$

$$=\frac{1}{2}\ln(m^2+y^2)\Big|_{y=a}^{y=b}=\frac{1}{2}\ln\left(\frac{m^2+b^2}{m^2+a^2}\right)$$

③ $H(a,b) = \int_0^\infty \frac{\ln(a^2+x^2)}{b^2+x^2}\,dx$

$\underline{H'_a} = 2a \int_0^\infty \frac{1}{(a^2+x^2)(b^2+x^2)}\,dx \; = $

$\left| \frac{1}{(a^2+x^2)(b^2+y^2)} = \frac{A}{a^2+x^2} + \frac{B}{b^2+x^2} \Rightarrow \ldots \right|$

$(a,b>0)$

$= \frac{2a}{b^2-a^2} \int_0^\infty \left(\frac{1}{a^2+x^2} - \frac{1}{b^2+x^2} \right) dx =$

$= \frac{2a}{b^2-a^2} \left(\frac{\pi}{2a} - \frac{\pi}{2b} \right) = \frac{\pi a}{b^2-a^2} \, \frac{b-a}{ab} =$

$= \pi \frac{b-a}{b(b^2-a^2)} = \underline{\pi \frac{1}{b(b+a)}}$

takže:

$\underline{\underline{H}} = \pi \int da \, \frac{1}{b(b+a)} = \underline{\underline{\frac{\pi}{b} \ln(b+a) + \varphi(b)}}$

ako poč. podmienku vyppoužijeme:

$H(a,a) = \int_0^\infty \frac{\ln(a^2+x^2)}{a^2+x^2}\,dx = \left| \begin{matrix} a>0 \\ x=ay \end{matrix} \right|$

$= \int_0^\infty \frac{\ln[a^2(1+y^2)]\,a\,dy}{a^2(1+y^2)} = \frac{1}{a} \int_0^\infty \left(\frac{\ln a^2}{1+y^2} + \frac{\ln(1+y^2)}{1+y^2} \right) dy =$

$= \frac{1}{a} \frac{\pi}{2} \ln(a^2) + \frac{1}{a} \underbrace{\int_0^\infty \frac{\ln(1+y^2)}{1+y^2}\,dy}_{= I}$

$$I = \int_0^\infty \frac{\ln(1+y^2)}{1+y^2}\,dy = \left| \begin{array}{l} y = \mathrm{tg}\, t \\ t = \arctan(y) \end{array} \right| =$$

$$= \int_0^{\pi/2} \frac{\ln(1+\mathrm{tg}^2 t)}{1}\,dt = \int_0^{\pi/2} \ln\left(\frac{\cos^2 t+\sin^2 t}{\cos^2 t}\right) dt =$$

$$= -2\int_0^{\pi/2} \ln(\cos t)\,dt = -2 \underbrace{\int_0^{\pi/2}\ln(\sin t)\,dt}_{J}$$

$$J = \int_0^{\pi/2}\ln(\sin x)\,dx = \left| x = 2y \right| = 2\int_0^{\pi/4}\ln(\sin 2y)\,dy =$$

$$= 2\int_0^{\pi/4}\left[\ln(2)+\ln(\sin y)+\ln(\cos y)\right]dy =$$

$$= \frac{\pi}{2}\ln(2) + 2\int_0^{\pi/4}\ln(\sin(y))\,dy + 2\int_0^{\pi/4}\ln(\cos(y))\,dy$$

$$\uparrow z = \frac{\pi}{2} - y$$

$$= \frac{\pi}{2}\ln(2) + 2\int_0^{\pi/4}\ln(\sin y)\,dy + 2\int_{\pi/4}^{\pi/2}\ln(\sin y)\,dy =$$

$$= \frac{\pi}{2}\ln(2) + 2J \Rightarrow$$

$$\Rightarrow \boxed{J = -\frac{\pi}{2}\ln(2)}$$

takže:

$$\boxed{I = -2J = \pi\ln(2)}$$

takže:

$$H(a,a) = \frac{\pi}{a}\ln(a) + \frac{1}{a}I = \frac{\pi}{a}\ln(a) + \frac{\pi}{a}\ln(2) =$$

$$= \boxed{\frac{\pi}{a}\ln(2a) \overset{!}{=} \frac{\pi}{a}\ln(2a) + \eta(a)} \Rightarrow$$

$\Rightarrow$ $\underline{\eta(a) = 0}$ a teda:

$$\boxed{H(a,b) \underset{a,b>0}{=} \frac{\pi}{b}\ln(a+b)}$$