$$R = \int_0^1 \ln \Gamma(x) \mathrm{d}x$$

$$R = \int_0^1 \ln \Gamma(x) \mathrm{d}x = [y = 1 - x] = -\int_1^0 \ln \Gamma(1 - y) \mathrm{d}y = \int_0^1 \ln \Gamma(1 - x) \mathrm{d}x$$

$$2R = \int_0^1 \mathrm{d}x \left\{ \ln \Gamma(x) + \ln \Gamma(1 - x) \right\} = \int_0^1 \mathrm{d}x \ln[\Gamma(x)\Gamma(1{-}x)] = \int_0^1 \mathrm{d}x \ln \frac{\pi}{\sin(\pi x)}$$

$$2R - \ln \pi = -\int_0^1 \mathrm{d}x \ln \sin(\pi x) = -Q$$

$$Q = \int_0^1 \mathrm{d}x \ln \sin(\pi x) = \int_0^{1/2} \mathrm{d}x \ln \sin(\pi x) + \int_{1/2}^1 \mathrm{d}x \ln \sin(\pi x)$$

$$\int_{1/2}^1 \mathrm{d}x \ln \sin(\pi x) = \left[y = x - \frac{1}{2} \right] = \int_0^{1/2} \mathrm{d}y \ln \sin \pi \left(y + \frac{1}{2} \right) = \int_0^{1/2} \mathrm{d}y \ln \cos(\pi y)$$

$$Q = \int_0^{1/2} \mathrm{d}x \ln \sin(\pi x) + \int_0^{1/2} \mathrm{d}x \ln \cos(\pi x) = \int_0^{1/2} \mathrm{d}x \ln[\sin(\pi x) \cos(\pi x)]$$

$$Q = \int_0^{1/2} \mathrm{d}x \ln \left[\frac{1}{2} \sin(2\pi x) \right] = -\frac{1}{2} \ln 2 + \int_0^{1/2} \mathrm{d}x \ln \sin(2\pi x) = -\frac{1}{2} \ln 2 + T$$

$$T = \int_0^{1/2} \mathrm{d}x \ln \sin(2\pi x) = [y = 2x] = \frac{1}{2} \int_0^1 \mathrm{d}y \ln \sin(\pi y) = \frac{1}{2} Q$$

$$Q = -\frac{1}{2} \ln 2 + \frac{1}{2} Q$$

$$Q = -\ln 2$$

$$R = \frac{1}{2} [\ln \pi - Q] = \frac{1}{2} [\ln \pi + \ln 2] = \ln \sqrt{2\pi}$$