Let us consider the function $F_4:\ \mathbb{C}\to\mathbb{C}$:

$$z\neq 0:\ F_4(z)=e^{-\frac{1}{z^4}},\ F_4(0)=0.$$

Let $z=x+iy$ and $F_4=U_4+iV_4$.

$$\frac{1}{z^4}=\frac{1}{x^4+y^4-6x^2y^2+i(4x^3y-4xy^3)}=\frac{x^4+y^4-6x^2y^2-i(4x^3y-4xy^3)}{(x^4+y^4-6x^2y^2)^2+16(x^3y-xy^3)^2}=$$
$$=\frac{x^4+y^4-6x^2y^2-i(4x^3y-4xy^3)}{(x^2+y^2)^4}=$$
$$=\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^2}-4ixy\frac{x^2-y^2}{(x^4+y^4)^2}.$$

$$U_4=\exp\left[\frac{-x^4-y^4+6x^2y^2}{(x^2+y^2)^4}\right]\cos\left[4xy\frac{x^2-y^2}{(x^2+y^2)^4}\right],$$
$$V_4=\exp\left[\frac{-x^4-y^4+6x^2y^2}{(x^2+y^2)^4}\right]\sin\left[4xy\frac{x^2-y^2}{(x^2+y^2)^4}\right].$$

Of course, for all $x,y:\ x^2+y^2>0$ functions $U_4,V_4$ obey Cauchy-Riemann equations. Moreover,

$$\frac{U_4(x,0)-U_4(0,0)}{x}=\frac{U_4(x,0)}{x}=\frac{1}{x}\exp[-\frac{1}{x^4}]\times 1\xrightarrow{x\to 0}0,$$

so there is finite partial derivative of $U_4$ w.r.t. $x$ at $(0,0)$ and equals zero. Similarly,

$$\frac{V_4(0,y)-V_4(0,0)}{x}=\frac{V_4(0,y)}{x}=0\xrightarrow{x\to 0}0,$$

and

$$\frac{U_4(0,y)-U_4(0,0)}{y}=\frac{U_4(0,y)}{y}=\frac{1}{y}\exp[-\frac{1}{y^4}]\times 1\xrightarrow{y\to 0}0,$$

so there is finite partial derivative of $V_4$ w.r.t. $y$ at $(0,0)$ and equals zero, and

$$\frac{V_4(x,0)-V_4(0,0)}{x}=\frac{V_4(x,0)}{x}=0\xrightarrow{x\to 0}0.$$

That means the real and imaginary parts of $F_4$ obey Cauchy-Riemann equations everywhere in $\mathbb{C}$.

On the other hand, for $t\in\mathbb{R}$:

$$F_4[\sqrt{i}t]=e^{-\frac{1}{-t^4}}\xrightarrow{t\to 0}\infty.$$

That means $F_4$ is unbounded in any neighborhood of 0 and therefore is not continuous at 0 and therefore is not differentiable at 0.

We can check the same for the function

$$z\neq 0:\ F_1(z)=e^{-\frac{1}{z}},\ F_1(0)=0.$$

Of course, $F_1$ is differentiable at any point different from 0. Separating $F_1$ into real and imaginary parts for $z\neq 0$ we obtain

$$F_1(z)=e^{-\frac{x-iy}{x^2+y^2}}=e^{-\frac{x}{x^2+y^2}}\cos\frac{y}{x^2+y^2}+ie^{-\frac{x}{x^2+y^2}}\sin\frac{y}{x^2+y^2}\equiv U_1+iV_1.$$

It is easy to check that:

- $$\frac{\partial U_1}{\partial x}(0,0)=0,\ \frac{\partial V_1}{\partial x}(0,0)=0,$$
- but partial derivatives of functions $U_1,V_1$ at $(0,0)$ do not exist.

So it is not surprising that $F_1$ is not differentiable at zero.

**Homework:** Verify the (non)differentiability vs. Cauchy-Riemann equations for the functions $F_2,F_3$ (at zero).

## References

[1] Looman, H. (1923), Über die Cauchy - Riemannschen Differentialgleichungen, Göttinger Nachrichten: 97 - 108.