Stirlingov vzorec

pre $|x|<$ poznáme:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - + \dots$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \right)$$

preto

$$(*) \ln \frac{1+x}{1-x} = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \quad , \quad |x|<1$$

vezmeme $n \in \mathbb{N}$ a $x = \frac{1}{2n+1}$

$$\frac{1+x}{1-x} = \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} = \frac{n+1}{n}$$

a podľa $(*)$

$$\ln\left(\frac{n+1}{n}\right) = 2 \cdot \frac{1}{2n+1} \left[1 + \frac{1}{3} \frac{1}{(2n+1)^2} + \frac{1}{5} \frac{1}{(2n+1)^4} + \dots \right]$$

alebo:

$$\left(n+\frac{1}{2}\right) \ln\left(1+\frac{1}{n}\right) = 1 + \frac{1}{3} \frac{1}{(2n+1)^2} + \frac{1}{5} \frac{1}{(2n+1)^4} + \dots \equiv \varphi(n)$$

zrejme:

$$1 \leq \varphi(n) < 1 + \frac{1}{3} \left[\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \dots \right] =$$

$$= 1 + \frac{1}{3} \frac{\frac{1}{(2n+1)^2}}{1 - \frac{1}{(2n+1)^2}} = 1 + \frac{1}{3} \frac{1}{(2n+1)^2 - 1} =$$

$$= 1 + \frac{1}{3} \frac{1}{4n^2+4n} = 1 + \frac{1}{12n(n+1)}$$

čiže $\forall n \in \mathbb{N}$ je

$$1 < \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) < 1 + \frac{1}{12n(n+1)}$$

a teda aj:

$$(+) \quad e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}$$

vezmime

$$a_n = \frac{n!\, e^n}{n^{n+\frac{1}{2}}}$$

potom

$$\frac{a_n}{a_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}}{e}$$

a z nerovnosti (+) máme:

$$1 < \frac{a_n}{a_{n+1}} < e^{\frac{1}{12n(n+1)}} = \frac{e^{\frac{1}{12n}}}{e^{\frac{1}{12(n+1)}}}$$

t.j. z jednej strany je $a_n > a_{n+1}$ a z druhej

$$a_n \cdot e^{-\frac{1}{12n}} < a_{n+1}\, e^{-\frac{1}{12(n+1)}}$$

t.j. a_n klesá s rastom n a $\{a_n\}$ je ohraničená zdola, t.j. $\exists\ a = \lim_{n\to\infty} a_n$. Z druhej strany $a_n \cdot e^{-\frac{1}{12n}}$ rastie a keďže $e^{-\frac{1}{12n}} \xrightarrow{n\to\infty} 1$, tak $a_n e^{-\frac{1}{12n}} \xrightarrow{n\to\infty} a$, pričom

$$a_n e^{-\frac{1}{12n}} < a < a_n$$

Existuje teda číslo θ (závislé od n) $0 < \theta < 1$ že:

$$a_n = a\, e^{\frac{\theta}{12n}}, \quad \text{t.j.} \quad \frac{n!\, e^n}{n^{n+\frac{1}{2}}} = a\, e^{\frac{\theta}{12n}},$$

alebo aj:

$$\text{(V)} \quad n! = a\sqrt{n}\left(\frac{n}{e}\right)^n e^{\frac{\theta}{12n}} \quad (0<\theta<1)$$

Stanovíme ešte "a":

Vieme elementárne, že:

$$\bullet \int_0^{\pi/2} \sin^{2n} x\, dx = \frac{(2n-1)!!}{(2n)!!}\,\frac{\pi}{2}$$

$$\bullet \int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{(2n)!!}{(2n+1)!!} \Rightarrow \int_0^{\pi/2} \sin^{2n-1} x\, dx = \frac{(2n-2)!!}{(2n-1)!!}$$

a máme nerovnosti

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx < \int_0^{\pi/2} \sin^{2n} x\, dx < \int_0^{\pi/2} \sin^{2n-1} x\, dx$$

t.j.

$$\frac{(2n)!!}{(2n+1)!!} < \frac{(2n-1)!!}{(2n)!!}\,\frac{\pi}{2} < \frac{(2n-2)!!}{(2n-1)!!}$$

$$\Rightarrow A_n = \frac{[(2n)!!]^2}{[(2n-1)!!]^2}\,\frac{1}{2n+1} < \frac{\pi}{2} < \frac{[(2n)!!]^2}{[(2n-1)!!]^2}\,\frac{1}{2n} = B_n$$

keďže $|A_n - B_n| = \frac{1}{2n(2n+1)}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 < \frac{1}{2n}\,\frac{\pi}{2} \xrightarrow{n\to\infty} 0$

tak máme (Wallisov vzorec):

$$\boxed{\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1}}$$

Formulu (♡) prepíšeme pre $n! \to (2n)!$:

$$(♡')\quad (2n)! = a\sqrt{2n}\left(\frac{2n}{e}\right)^{2n} e^{\frac{\theta'}{24n}} \qquad (0<\theta'<1)$$

teraz: $$(2n)! = \frac{2^{2n}(n!)^2}{(2n)!!}(2n-1)!!$$

t.j.

$$\frac{2^{2n}(n!)^2}{(2n)!!}(2n-1)!! = a\sqrt{2n}\left(\frac{2n}{e}\right)^{2n} e^{\frac{\theta'}{24n}}$$

alebo aj $n!$ zapíšem z (♡) a máme:

$$2^{2n}\frac{(2n-1)!!}{(2n)!!}\, a^2\cdot n\left(\frac{n}{e}\right)^{2n} e^{\frac{2\theta}{12n}} = a\sqrt{2n}\left(\frac{2n}{e}\right)^{2n} e^{\frac{\theta'}{24n}}$$

$$a\cdot e^{\frac{4\theta-\theta'}{24n}}\cdot\sqrt{\frac{n}{2}} = \frac{(2n)!!}{(2n-1)!!}$$

$$a^2 = \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{2}{n}\, e^{-\frac{4\theta-\theta'}{24n}} \xrightarrow{n\to\infty} 4\cdot\frac{\pi}{2} = 2\pi$$

$$\boxed{a = \sqrt{2\pi}}$$

Finálne máme z (♡) tzv. STIRLINGOV VZOREC:

$$\boxed{n! = (2\pi n)^{\frac{1}{2}}\left(\frac{n}{e}\right)^{n} e^{\frac{\theta}{12n}} \qquad (0<\theta<1)}$$